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NONFORWARD SUPERCONVERGENCE RELATIONS

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A superconvergence relation for $\pi\rho$ scattering is saturated for a range of t with an infinite number of mesons of all spins and nondegenerate masses.

Recently Fubini derived the SU(6) results $g_{\varphi\rho\pi}=0$ and $g_{\rho\pi\pi}^2=\frac{1}{2}g_{\omega\rho\pi}^2M^2$ by saturating two superconvergent sum rules (SCR) for $\pi\rho$ scattering at $t=0$.[1] Fubini has also shown two ways by which an SCR could be saturated for a range of t without using states of isospin two: (1) The SCR can be saturated with a finite number of particles, but some coupling constants are forced to be imaginary; (2) the SCR can be saturated with an infinite number of particles (a tower), but they must be degenerate in mass.[2] We shall show that the SCR can be saturated for a range of t by using a tower of mesons not necessarily degenerate in mass.

As the saturating particles may have high spin, it is convenient to use helicity amplitudes. $T_{bb',aa'}{}^I(s,t)$ is the helicity amplitude for the t-channel process $(A+A'\to B+B')$. The superscript labels isospin, and the subscripts label helicity. $\overline{T}_{bb',aa'}{}^I(s,t)$ is the amplitude free of kinematical singularities as given by Wang.[3] Similarly $S_{a'b',ab}{}^I(s,t)$ is for the s-channel process $(A+B\to A'+B')$.

If we allow ourselves to treat ρ as a stable particle and if $T_{aa'}{}^1(s,t)/s\to 0$ as $s\to\infty$ for the reaction $\rho+\rho\to\pi+\pi$, then the fixed-t dispersion relation for $\overline{T}_{+-}{}^1(s,t)$ can be converted into the SCR[4]

$$\int_{M_\pi^2}^{\infty}\mathrm{Im}\overline{T}_{+-}{}^1(s',t)ds'=0. \tag{1}$$

Equation (1) corresponds to the SCR for the invariant amplitude $A(s,t)$ used by Fubini.[1] The SCR which Fubini writes for $\overline{T}_{+-}{}^2(s,0)$, corresponding to the invariant amplitude $B(s,t)$, may not be correct because of the presence of Regge cuts,[5] and will not be considered in this note.

The amplitude $\overline{T}_{+-}{}^1(s,t)$ can be related to $S_{a,a'}{}^I(s,t)$ by using the helicity crossing matrices[6] and the isospin crossing matrix $C_{II'}$:

$$\overline{T}_{+-}{}^1(s,t)\equiv T_{+-}{}^1(s,t)[(1-x_t^2)(t-4\mu^2)]^{-1}=\sum_{a,a',I}(-)^{a'}C_{1I}S_{a',a}{}^I(s,t)\frac{d_{a',+}{}^1(x_A)d_{a,+}{}^1(x_A)M^2}{4sq^2(1-x_1^2)}, \tag{2}$$

$$M=M_\rho,\quad \mu=M_\pi,\quad x_s=1+t/2q^2,\quad x_1=\left[\frac{t}{t-4M^2}\right]^{1/2}\frac{E}{q},\quad 1-x_t^2=\frac{(1-x_1^2)4sq^2}{M^2(t-4\mu^2)},$$

where E, q are the energy and momentum of the ρ in the s channel. We have used $d_{a',-}{}^{J}(x_A) = (-)^{J-a'}d_{a',+}{}^{J}(x_A)$.[6]

For t in a neighborhood of $t=0$, a partial-wave decomposition of S can be made:

$$C_{1I}S_{a',a}{}^{I}(s,t) = \sum_J (2J+1) f_{a',a}{}^{JI}(s) d_{a',a}{}^{J}(x_s). \quad (3)$$

We shall now show that the SCR can be satisfied for a range of t with the following saturation scheme. We assume that each partial wave is dominated by a single zero-width resonance. The particles with odd spin have an isospin of zero and odd parity (ω-meson trajectory), and the particles with even spin have an isospin of one and even parity (A_2 trajectory) except for the case of spin 0 which has odd parity (π). We can therefore drop the isospin superscript, and since $f_{a',a}{}^{J}(s) = -f_{a',-a}{}^{J}(s)$ for states of parity $(-)^J$, we can drop the helicity subscripts as well.

The SCR can now be written as

$$0 = 2a_0 M^2 + \sum_{J=1} [a_J P_J'(x_s)(t-2M^2) + P_J''(x_s)(t-4M^2)(t/2q_J{}^2)], \quad (4)$$

with[7]

$$sq^2 a_J \delta(s-M_J{}^2) = (2J+1)\,\mathrm{Im} f_{1,1}{}^{J}(s) > 0 \text{ for } J>0,$$

$$sq^2 a_0 \delta(s-\mu^2) = \mathrm{Im} f_{0,0}{}^{0}(s) > 0,$$

and P_J' is the derivative of a Legendre polynomial. To have the SCR true for a range of t, the coefficient of t^n must be equal to 0, which gives

$$0 = \sum_{J=n}^{\infty} \frac{a_J}{q_J{}^{2n}} G_n(J) \frac{(J+n)!}{(J-n)!} \text{ for } n \geq 2, \quad (5)$$

where $G_n(J) = J^2 - (E_J n/M)^2 + J - n$ and $E_J = (M_J{}^2 + M^2 - \mu^2)/(2M_J)$.

In order to simplify the solution of (5) it is convenient to define $b_J = a_J[(E_J+M)/(E_J-M)]^J$ and put the SCR in the form

$$0 = \sum_n b_J F_n(J)(J-\bar{J}),$$

$$F_n(J) = \frac{(E_J-M)^{J-n}(J+n)!\,G_n(J)}{(E_J+M)^{J+n}(J-n)!\,(J-\bar{J})} > 0, \quad (6)$$

where $\bar{J}$ is the solution of $G_n(\bar{J}) = 0$. If we make the reasonable assumption that E_J is an increasing function of J, then $\bar{J}$ is a unique function of n.

The whole problem is now reduced to the following question. Can a function $b_J > 0$ be found such that the partial summation in (6) with $J < \bar{J}$ cancels the summation with $J > \bar{J}$, for all n?

In order to determine b_J, we shall restrict the summation to an interval symmetric about $\bar{J}$, such as $3\bar{J}/4 < J < 5\bar{J}/4$. Our neglect of the tails can be justified for large n, since the function $F_n(J)$ is strongly peaked at $\bar{J}$. In fact, our desire to have $F_n(J)$ peaked at $\bar{J}$ was the motivation behind our definition of b_J. Near $J = \bar{J}$ we have

$$F_n(J) \approx \left(\frac{MJ}{E_J}\right)^{2n} \exp\left(-\frac{MJ}{E_J}\right),$$

$$\frac{F_n(J)}{F_n(\bar{J})} \approx [(1+x)e^{-x}]^{2n} \approx \exp(-x^2 n), \quad (7)$$

where $1 + x = JE_{\bar{J}}/\bar{J}E_J$.

The sharpness of the peaking depends on the shape of the mass spectrum. For E_J/M proportional to J^β, we immediately see from (5) that β must be less than 1, in order to have the possibility of a solution. As β approaches 1, we see from (7) that the minimum value of n needed to justify dropping the tails becomes very large. We shall therefore be unable to find an explicit expansion for b_J when J is small.

The inability to determine b_J when J is small does not invalidate our procedure, for the hardest task is to satisfy the infinite number of SCR given by (6) for large n. One can then determine b_J for smaller J by explicitly summing over those values of J for which b_J had previously calculated.

For n sufficiently large, we can expand (6) and get

$$0 = \sum_{J=3\bar{J}/4}^{5\bar{J}/4} \left\{ [b_J F_n(J)]_{J=\bar{J}}{}'(J-\bar{J})^2 + [b_J F_n(J)]_{J=\bar{J}}{}''' \frac{(J-\bar{J})^4}{3!} + \cdots \right\}. \quad (8)$$

The odd terms of the Taylor's expansion are not present since the summation interval is symmetric around $\bar{J}$. In the present first-order calculation, only the first term in (8) will be kept, as the higher terms turn out to be smaller by a factor of $(1/J)^2$.

We now set $[b_J F_n(J)]_{J=\bar{J}}{}' = 0$ for a fixed n and then express n in terms of $\bar{J}$ to get an equation for b_J independent of n:

$$\left.\frac{b_J'}{b_J}\right|_{J=\bar{J}} = -\left.\frac{F_n'(J)}{F_n(J)}\right|_{J=\bar{J}} = \left\{\frac{M}{E_{\bar{J}}+M}\left(\frac{E_{\bar{J}}'}{2E_{\bar{J}}}+\frac{2E_{\bar{J}}-M}{\bar{J}(E_{\bar{J}}-M)}\right)-\frac{1}{2}\left[\frac{1}{\bar{J}}+\frac{E_{\bar{J}}'}{E_{\bar{J}}}+\frac{\bar{J}E_{\bar{J}}''}{\bar{J}E_{\bar{J}}'-E_{\bar{J}}}\right]\right\}[1+O(1/J)]. \tag{9}$$

The term in square brackets is dominant and can be explicitly integrated to give

$$b_J \propto \frac{M}{E_J J}\left(\frac{E_J'}{E_J}-\frac{1}{J}\right)^{-1/2}\left[1+O\left(\frac{M}{E}\right)\right]. \tag{10}$$

To determine the higher order corrections we must specify the function E_J in order to integrate (9). However, only the corrections up to $O(1/J^2)$ are straightforward to calculate. To continue the expansion of b_J beyond $O(1/J^2)$, we must include the terms of (8) which have been neglected, whereupon the analysis becomes much more complicated.

We have indicated how one can construct an explicit solution of the SCR once the mass spectrum of the saturating particles is given. It is indeed interesting to note that our result for the dominant behavior of a_J agrees with a calculation which Fubini made for the case of degenerate masses.[8]

Our approach to saturating the SCR may be useful for Gell-Mann's program of finding a representation of the current plus angular momentum algebra.[9] Gell-Mann's program requires that the particles used in the intermediate states must also give valid SCR when used as external particles. That is, we must also saturate the SCR for $\pi+\rho\to\pi+A_2$, $\pi+A_2\to\pi+A_2$, $\cdots$. It will surely be difficult to saturate the complete set of SCR, but the helicity techniques presented in this paper may be the appropriate formalism with which to attack this problem.

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