[14]This is a less specific statement than given in Paper I below Eq. (30) and is made in the light of all the data now available. We thank Dr. E. Bloom and Dr. R. Taylor of the Stanford Linear Accelerator Center for discussions of the data in its present state.

[15]See the discussion below Eq. (30) in Paper I.

FERMION REGGEIZATION WITHOUT PARITY DOUBLING*

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The common belief that fermions lying on linear trajectories must have opposite-parity partners is shown to be false. Reggeization of a sequence of positive-parity fermion resonances is carried out in the Van Hove model. As a consequence of the absence of negative-parity states, the partial-wave amplitudes must have a fixed cut in the J plane. This fixed cut, in conjunction with the moving Regge pole, provides a new parametrization for fermion-exchange reactions, which is in qualitative agreement with the data.

Gribov[1] showed that every fermion Regge trajectory $[\alpha^+(W)]$ must be accompanied by a MacDowell symmetric[2] trajectory $[\alpha^-(W)=\alpha^+(-W)]$ of the opposite parity. If (as is indicated by experiment for N_α and Δ_δ) a trajectory is linear in $u=W^2$, its MacDowell twin will be degenerate with it. Hence it has always seemed puzzling that no parity partners of the N and $\Delta(1238)$ have been found. Attempts to find an analytic form in which states on the MacDowell twin are systematically suppressed have not been successful.[3] We deduce the appropriate analytic form from a model containing only resonances of positive parity lying on a linear trajectory. The partial-wave amplitudes are found to have a fixed Regge cut, and the negative-parity (MacDowell twin) trajectory lies on an unphysical sheet of the J plane at positive energies. The idea of a fixed Regge cut is not new; it is present in the solution of the Dirac equation with a Coulomb potential.[4] In the present problem it is, of course, possible to have parity doubling and no Regge cut; but lacking any a priori reason for parity doubling, we anticipate in general the presence of a fixed Regge cut in fermion-exchange amplitudes.

We will illustrate the origin of the fixed cut in the Van Hove model.[5] The amplitude in this model is the sum of Feynman diagrams for the exchange of all resonances along a given trajectory. Clearly, this amplitude satisfies the usual analyticity requirements and contains only the resonances of the input trajectory. In πN scattering, the Feynman diagram for the exchange of a natural-parity ($J^P=\frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+, \cdots$) fermion resonance of spin $J=l+\frac{1}{2}$ and mass $m(l)$ in the u channel[6,7] is

$$\begin{aligned}\bar{u}_2\mathfrak{M}(J)u_1 &= \bar{u}_2\gamma_5 g^2(l)p_{\mu_1}'\cdots p_{\mu_l}'T_{\mu_1\cdots\mu_l;\nu_1\cdots\nu_l}{}^J p_{\nu_1}\cdots p_{\nu_l}\gamma_5 u_1\\ &= \bar{u}_2\frac{g^2(l)p^{2l}P_{l+1}'(z_u)}{u-m^2(l)}\left[\frac{\not{k}}{m(l)}-1\right]u_1+O(z_u{}^{l-1}),\end{aligned} \tag{1}$$

where $T_{\mu,\nu}{}^J$ is the propagator for a spin-J fermion. We Reggeize by summing a sequence of resonances and transforming the sum into an integral a la Sommerfeld and Watson[8]:

$$\mathfrak{M}=\sum_J\mathfrak{M}(J)\simeq\frac{i}{2}\int dl\frac{g^2(l)p^{2l}P_{l+1}(-z_u)}{[u-m^2(l)]\sin\pi l}\left[\frac{\not{k}}{m(l)}-1\right]. \tag{2}$$

All terms but those contributing to the leading power of the asymptotic expansion of $\mathfrak{M}(u,z_u)$ as $z_u\to\infty$ have been dropped.

If we take $m^2(l)=(l-\alpha_0)/\alpha'$ and assume for convenience that $g^2(l)$ is analytic in l,[9] we can open the contour in the l plane and obtain a contribution from the pole at $m^2(l)=u$ and the cut with branch point at $l=\alpha_0$ (see Fig. 1). This gives

$$\mathfrak{M}(u,z_u)=\frac{\pi g^2(\alpha(u))p^{2\alpha(u)}P_{\alpha(u)+1}'(-z_u)\alpha'}{\sin\pi\alpha(u)}\left[\frac{\not{k}-W}{W}\right]-\not{k}\int_{-\infty}^{\alpha_0}dl\frac{g^2(l)p^{2l}P_{l+1}'(-z_u)}{[-m^2(l)]^{1/2}[u-m^2(l)]\sin\pi l}, \tag{3}$$

where

$$\alpha(u)=\alpha_0+\alpha'u. \tag{4}$$

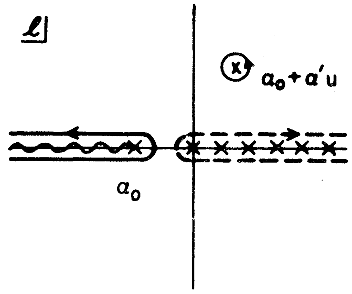

FIG. 1. Dashed line, initial contour; solid line, opened contour.

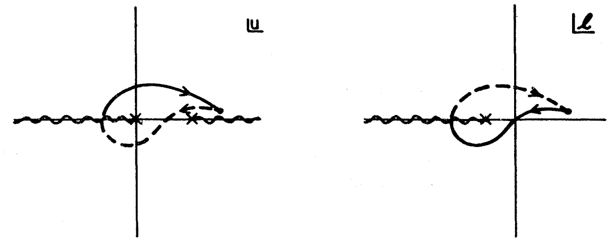

FIG. 2. Pole trajectory for f^+. Dashed lines show path on second sheet; solid lines refer to the principal sheet.

In the limit $z_u \to \infty$,

$$\mathfrak{M}(u,z_u)\to\frac{\pi g^2(\alpha(u))c(\alpha(u))s^{\alpha(u)}\alpha'}{\sin\pi\alpha(u)}\left[\frac{\not k-W}{W}\right]-\not k\int_{-\infty}^{\alpha_0}dl\frac{g^2(l)c(l)s^l}{[-m^2(l)]^{1/2}[u-m^2(l)]\sin\pi l},\tag{5}$$

where

$$c(l)=\frac{\Gamma(2l+2)}{4^l[\Gamma(l+1)]^2}.\tag{5}$$

The first term of (5) has the form of a Regge-pole contribution while the second term has the form of a Regge cut. The singularity at $u=0$ in the residue of the pole term is cancelled by the cut term, so that $\mathfrak{M}$ is analytic in u [see Eq. (11)].

The principal features of our solution can be seen in the partial-wave amplitudes $f_{J\pm 1/2}{}^{\mp}(W)$,[2] which can be read off directly from (1). We find that

$$f_{J\pm 1/2}{}^{\mp}=-\frac{E\mp M}{8\pi W}\left(\frac{\alpha'^{1/2}W\pm(l-\alpha_0)^{1/2}}{l-\alpha_0-\alpha' u}\right)\frac{\alpha'}{(l-\alpha_0)^{1/2}}p^{2l}g^2(l),\tag{7}$$

where $l=J-\frac{1}{2}$. There is a moving pole at $l=\alpha(u)$ and a fixed cut at $l=\alpha_0$. Note that the moving pole in $f_{J-1/2}{}^{+}$ is on the physical sheet of the l plane only for $W<0$. As we move from $W<0$ to $W>0$, the pole at $l=\alpha(W^2)$ moves through the fixed cut onto the second sheet of the l plane as shown in Fig. 2; this explains why there are no negative-parity resonances in our model.

The cut $(l-\alpha_0)^{-1/2}$ in (7) is due to the presence of odd powers of $m(l)$ in (1). We can obtain a solution with no Regge cut only if we include negative-parity states along with the positive-parity ones.[10] This would correspond to the usual solution; but it clearly involves the ad hoc assumption that the negative-parity states exist.

Unless $g^2(\alpha_0)=0$, the partial-wave amplitudes (7) have infinities for $l\to\alpha_0$, in violation of unitarity. Of course, our model has zero-width resonances, so it is clearly not unitary. We wish to demonstrate that there exists a smooth limit from the unitarized theory to the zero-width limit, and that this limit should be useful in parametrizing experimental data. The procedure for unitarizing the model has been discussed by Sugar and Sullivan,[11] who find that certain fixed poles are converted into moving poles in the process.

In our case[12] the unitarized partial-wave amplitudes have the form

$$f_{J\pm 1/2}{}^{\mp}=\frac{E\pm M}{8\pi W}\frac{p^{2l}g^2(l)}{[m(l)-g^2(l)a(u)]W-[m^2(l)+g^2(l)b(u)]},\tag{8}$$

where $a(u)$ and $b(u)$ are functions with the proper right-hand cuts in u. The amplitudes $f_{J\pm 1/2}{}^{\mp}$ have a fixed cut at $l=\alpha_0$ and only moving poles. In particular if $g^2(l)=c$, a constant, $f_{J\pm 1/2}{}^{\mp}$ have two moving poles,[13] with trajectories $\alpha_{1,2}$ [14] given by

$$2m(\alpha_{1,2}(u))=W\pm[W^2-4cWa(u)-4cb(u)]^{1/2}.\tag{9}$$

In the limit $a,b\to 0$, $\alpha_1\to\alpha(u)$ (the input positive-parity trajectory) and $\alpha_2\to\alpha_0$ (a fixed pole). Thus we can interpret the factor $(l-\alpha_0)^{-1/2}$ in (7) as the coincidence of a fixed cut, $(l-\alpha_0)^{1/2}$, and a fixed pole,

$(l-\alpha_0)^{-1}$. When the model is unitarized, the fixed pole becomes a moving pole just as in Ref. 11. Although the pole at $l=\alpha_2(u)$ does contribute to the asymptotic scattering amplitude, as long as a and b are small the trajectory $\alpha_2(u)$ will never rise high enough to produce any physical resonances.

Unless $a(u)$ and $b(u)$ are small, the unitarized trajectory will deviate from the linear form (4). Since experiment indicates approximately linear trajectories, we conclude that a and b may be neglected and data may be parametrized using (5).

For small negative u, we make the approximation

$$\frac{g^2(\alpha(u))c(\alpha(u))\alpha'}{\sin\pi\alpha(u)} \simeq G_0+G_1u. \tag{10}$$

Then (5) becomes

$$M \simeq \pi\left(\frac{G_0}{W}+G_1W\right) s^{\alpha_0+\alpha' u}(\not{k}-W)-\pi\left[\left(\frac{G_0}{W}+G_1W\;\; s^{\alpha' u}\{1-\mathrm{erf}[(\alpha' u \ln s)^{1/2}]\}-\frac{G_1}{(\alpha'\pi \ln s)^{1/2}}\right] s^{\alpha_0}\not{k}. \tag{11}$$

The first term is clearly a Regge pole, and the second is a fixed Regge cut, since

$$\mathrm{erf}x \xrightarrow[|x|\to\infty]{} 1-\frac{1}{\sqrt{\pi}}e^{-x^2}\left[\frac{1}{x}-\frac{1}{2x^3}+\cdots\right] \quad \left(|\arg x|<\frac{3\pi}{4}\right). \tag{12}$$

We can see explicitly how the singularity in the pole residue is canceled by the cut. The remaining part of the cut term has no $\sqrt{u}$ singularity because $\mathrm{erf}x$ is odd in x. Signature may be incorporated in our formulas by the modification

$$\mathfrak{M}(u,z_u)\to\tfrac{1}{2}[\mathfrak{M}(u,z_u)+\tau\mathfrak{M}(u,-z_u)].$$

With this modification, the pole term in (11) will acquire the usual signature factor and the cut will have a complicated varying phase.

The strongest experimental support of our work lies in the absence of parity partners to known fermion resonances. Also our conclusion that the partial-wave amplitudes contain a fixed Regge cut does not clash with experiment. Note that by an appropriate choice of G_0 and G_1, the ratio of the cut contribution to that of the pole can be chosen arbitrarily for a given range of s. If the pole contribution is dominant, we expect to see typical Regge shrinkage and dips where the trajectory passes through wrong-signature nonsense points. When the cut dominates, there will be no shrinkage and no wrong-signature nonsense dips. Nucleon-exchange data support this correlation. In π^+p backward scattering, the data show Regge shrinkage and a marked dip at $u=-0.2$. In backward π^0 photoproduction, there is no shrinkage and no dip at $u=-0.2$.

We would like to thank Professor S. C. Frautschi for his advice and encouragement. After completing this work, we learned that M. Halpern and J. Mandula have also investigated the problem of eliminating parity doublets in fermion Reggeization.

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[1]V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 43, 152 (1962) [translation: Soviet Phys.-JETP 16, 1080 (1963)].

[2]S. W. MacDowell, Phys. Rev. 116, 774 (1959).

[3]V. Barger and D. Cline, Phys. Letters 26B, 85 (1967), suggest that only the lowest state on each MacDowell twin is suppressed. This *ad hoc* assumption has also been made in Veneziano models of πN scattering, which have failed to fit backward scattering data even qualitatively. See, e.g., S. Chu and B. R. Desai, University of California at Riverside Report No. UCR-34P107-96 (to be published).

[4]The fixed cut gives the trajectory functions a two-sheeted J-plane structure. In an analytic continuation from $E>0$ to $E<0$, the Regge poles move from one sheet of the J plane to the other. See V. Singh, Phys. Rev. 127, 632 (1962).

[5]L. Van Hove, Phys. Letters 24B, 183 (1967); R. P. Feynman, unpublished.

[6]The total momentum in the u channel is denoted by k_μ, and p_μ and p_μ' are the initial and final nucleon momenta, respectively. The energy, momentum, and scattering angle in the u-channel center of mass are W $(=\sqrt{u})$, p, and z_u.

[7]The factor $[\not{k}-m(l)]/m(l)$ corresponds to a normalization $\bar{u}_l u_l=2$, independent of l.

[8]Properly speaking, we should work with Legendre functions of the second kind to make an analytic continuation

to the left of $\mathrm{Re}\,l = -\frac{1}{2}$.

[9]In general, we expect $g^2(l)$ to be a meromorphic function of $m(l)$. Even powers of $m(l)$ yield a fixed cut in the spin-flip amplitude, while odd powers give a fixed cut in the nonflip amplitude. Nucleon exchange in backward π^+p scattering may be qualitatively fitted with only even powers of $m(l)$ in $g^2(l)$.

[10]The contribution of the Feynman diagram for the exchange of an unnatural parity resonance has the form of (1) with $m(l)$ replaced by $-m(l)$. $\mathfrak{M}(m(l)) + \mathfrak{M}(-m(l))$ has no odd powers of $m(l)$ and hence no cut in l.

[11]R. L. Sugar and J. D. Sullivan, Phys. Rev. 166, 1515 (1968).

[12]See D. L. Steele, thesis, University of Illinois, 1969 (unpublished).

[13]If $g^2(\alpha_0) = 0$, there would be no infinity in (7) and no auxiliary pole $\alpha_2(u)$. However, the Regge-pole contribution to the nonflip part of $\mathfrak{M}$ at $u=0$ would be of order $b(u)$, which by assumption is small. Nucleon-exchange data in backward π^+p scattering shows no indication of any dip at $u=0$. (See also Ref. 9.)

[14]Note that α_1 and α_2 collide at $u=0$ and become complex for $u<0$. If there is only one moving pole (see Ref. 13), then α does not become complex. Also note that $\alpha_{1,2}(0)$ no longer coincide with the branch point α_0.

SOFT-PHOTON THEOREMS AND RADIATIVE K_{l3} DECAYS*

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We apply the soft-photon theorems of Low, Adler and Dothan, and Burnett and Kroll to the radiative decays $K^- \to \pi^0 l^- \bar{\nu}\gamma$ and $\bar{K}^0 \to \pi^+ l^- \bar{\nu}\gamma$ ($l = e$ or μ) to obtain the leading terms in the respective matrix elements. Numerical results for the photon spectra and for the decay rate (as a function of the minimum photon energy) are given in terms of the conventional K_{l3} parameters $f_+(0)$, ξ, λ_+, and λ_-.

In this Letter we present the results of an extensive theoretical investigation of radiative K_{l3} decays. Because the ordinary K_{l3} decays have been studied in great detail, both experimentally and theoretically, the radiative modes provide a unique opportunity to check the predictions of soft-photon theorems, in particular the presence of derivative terms, which are not present in two-body decays and very difficult to observe in scattering processes due to the lack of a simple theory of the elastic-scattering matrix elements. A few of these radiative events have already been seen[1] and thus our results are of immediate interest to those phyicists working in this area. With slight modifications, experiments now in progress[1] could be designed to examine radiative K_{l3} events and check our theoretical predictions. We give all our results in terms of standard K_{l3} parameters $f_+(0)$, ξ, λ_+, and λ_-. For full details of the calculations, we refer the reader to a previous paper[2] and to another to be submitted for publication.[3]

Let us write down the relevant K_{l3} matrix element to establish our notation. Assuming the $|\Delta \vec{I}| = \frac{1}{2}$ rule, $V-A$ theory, and μ-e universality, we obtain the T-matrix element for $\bar{K}^0(P) \to \pi^+(Q) + l^-(p) + \bar{\nu}(q)$ (or $K^- \to \pi^0 l^- \bar{\nu}$):

$$T(K \to \pi l \nu) = \bar{u}(p)[f_+(t) i\gamma \cdot (P+Q) + f_-(t) i\gamma \cdot (P-Q)](1+\gamma_5)v(q), \tag{1}$$

where $t = -(P-Q)^2$. In the SU(3) limit $f_-(0) = 0$ and $f_+(0) = 1/\sqrt{2}$ or 1 for charged or neutral K decays, respectively, and are real as we neglect CP-nonconserving effects. The decay rates are

$$\Gamma(K^- \to \pi^0 e^- \bar{\nu}) = 4\Gamma_0 \sin^2\theta f_+^2(0)(1.1826 + 4.3725\lambda_+) \times 10^{-2}, \tag{2}$$

$$\Gamma(\bar{K}^0 \to \pi^+ e^- \bar{\nu}) = 4\Gamma_0 \sin^2\theta f_+^2(0)(1.1977 + 4.1396\lambda_+) \times 10^{-2}, \tag{3}$$

$$\Gamma(K^- \to \pi^0 \mu^- \bar{\nu}) = 4\Gamma_0 \sin^2\theta f_+^2(0)[0.7636 + 4.4925\lambda_+ + 0.0227\xi^2 + 0.1992\xi^2\lambda_- + 0.1495\xi + 0.5622\xi(\lambda_+ + \lambda_-)] \times 10^{-2}, \tag{4}$$

$$\Gamma(\bar{K}^0 \to \pi^+ \mu^- \bar{\nu}) = 4\Gamma_0 \sin^2\theta f_+^2(0)[0.7728 + 4.2474\lambda_+ + 0.0223\xi^2 + 0.1828\xi^2\lambda_- + 0.1492\xi + 0.5234\xi(\lambda_+ + \lambda_-)] \times 10^{-2}, \tag{5}$$

where $\xi = f_-(0)/f_+(0)$, θ is the Cabibbo angle, $\Gamma_0 = G^2\bar{M}^5/64\pi^3 = 3.118 \times 10^9\ \mathrm{sec}^{-1}$ with $G = 1.435 \times 10^{-49}$ erg cm^3 the Fermi constant obtained from muon decay and with $\bar{M} = \frac{1}{2}(M_{K^0} + M_{K^-})$, and where the pa-