Hyperacuity thresholds of 1 sec: theoretical predictions and empirical validation

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Bisection thresholds were measured as a function of the separation of the lines. For separations of less than 1.5 min, the addition of flanking lines facilitates bisection so that thresholds of less than 1 sec for discriminating the direction of offset could be reliably obtained. For larger separations an interval could be bisected to an accuracy of 1 part in 60. Experiments varying the length, luminance, and overlap of the lines suggest that different cues are used in these two regimes. A dual space–size analysis is presented that can account for these bisection thresholds over a wide range of experimental conditions. This quantitative analysis produces viewprints of the stimuli (analogous to the voiceprint of audition). Each viewprint shows the output of many spatial filters of different positions and sizes. A new filter shape is introduced that has advantages for modeling the visual system. The sensitivity of each filter is fixed by the contrast-response function. The analysis further shows that the limiting factors in spatial hyperacuity are both the contrast-response function and the spatial grain.

INTRODUCTION

The human visual system is capable of judging relative position with remarkable accuracy. Thresholds for these tasks are often 3–5 arc sec. These low thresholds are 5–10 times finer than either the cutoff spatial frequency or the intercone spacing. For this reason, Westheimer has coined the term hyperacuity to describe a variety of tasks that involve sensing the direction of spatial offset of a line or point relative to a reference.

Just how the human brain achieves this remarkable accuracy for relative position has been a subject of much recent experimentation and speculation. The present paper has three main purposes. First, the results of several bisection experiments are presented. Under optimal conditions, the observers could correctly discriminate the relative position of a line with an accuracy of less than 1 arc sec. This acuity value is considerably better than previously reported hyperacuities, so the particular conditions resulting in these low thresholds are of interest. Second, a quantitative methodology for sensing fine relative position, based on mechanisms localized in space and size, is developed. Finally, the method is applied to understanding the present hyperacuity experiments, and links among spatial frequency, contrast sensitivity, and hyperacuity are discussed.

METHODS AND STIMULI

The experiments involve a bisection task. A schematic of one of the stimuli (near the optimal spacing of 1.3 min) is shown in Fig. 1. The stimuli consisted of bright (0.56 cd/m\(^2\); see the section on calibration for details), thin, high-contrast horizontal lines, each 30 min long, generated on the cathode-ray tube of a computer (Commodore 2001 with green phosphor). The display was modified so that the vertical screen height could be shrunk, resulting in a pixel size of less than 2 sec when viewed from a distance of 10 m. The outer “reference” lines were presented continuously. A “test” line was flashed for either 0.6 sec (for observer SK) or 1 sec (for observer DL) in one of five positions equally spaced around the bisection point. Viewing was binocular with natural pupils and normal room illumination.

The observer’s task was to judge the position of the test line relative to the bisection point by giving numbers from −2 to +2. This self-paced method of constant stimuli with multiple responses was described previously. Thresholds for discriminating the position of the test line were obtained by calculating a maximum-likelihood estimate of the \(d'\) values for each stimulus and interpolating to \(d' = 0.675\), equivalent to the usual 75% correct level. This analysis is a multiple-criterion probit analysis. The analysis program, ROCFLEX, is available for Commodore computers on request.

Data were collected in blocks of 125 trials, with the spacing of the reference lines varying between blocks. The computer randomized the position of the test lines, tallied the responses, and provided feedback following each trial. The authors served as observers. Both had corrected-to-normal vision. Extensive practice was provided. The main results have been confirmed in several other observers.

CALIBRATION

Since the test lines were thinner (<14 sec) than the blur function of the eye, it is useful to specify each line’s luminance in units of candellas per meter (linear luminance) rather than candellas per square meter (surface luminance). Linear luminance, which equals the surface luminance times the width of the line, was calibrated as follows. The full cathode-ray tube display (200 vertical pixels) was turned on, and its luminance was measured to be 925 cd/m\(^2\) when the screen was shrunk to the 1.64-cm height that was the useful value used in our experiments. Each line of our stimulus was 7 pixels wide, so each line in principle had a height of (1.64 × 7/200) cm = 0.061 cm. Thus the linear luminance of a 7-pixel line was 0.56 cd/m. This calculation gives the correct linear luminance even though the blur characteristics of the cathode-ray tube

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results.

The reference lines were presented continuously, and the test line flashed on for 0.6 sec (for subject SK) or 1.0 sec (for subject DL) in one of five positions (0, 1, or 2 pixels above or below the bisection point). For most of the experiments each pixel subtended 1.64 sec at the 10-m testing distance.

EXPERIMENTAL RESULTS

The triangles in Fig. 2 show how the bisection threshold depends on the spacing between the lines. For separations greater than 1.3 min, thresholds increase more or less linearly with increasing separation. The line in Fig. 2 has a slope of 1, showing that threshold is a constant fraction (1/60) of the separation, i.e., a Weber fraction. This constant fraction of 1/60 means that for a 3-min separation between the test line and the reference lines there will be a 3-sec bisection threshold. This same Weber fraction is well known and appears to hold for separations as large as 20 deg.

At separations between 2 and 2.5 min, a surprising degradation of bisection acuity was found. In order to be assured that the degradation was not a statistical fluctuation, many counterbalanced runs were repeated at separations of 1.75, 2.0, 2.25, and 2.5 min, giving 750 trials at each separation. Figure 2 shows that the threshold at separations of 2.0 and 2.25 min is higher than at 1.75 or 2.5 min. Since this spurious peak seemed so peculiar, the experiment was redone with a modified set of stimuli. Rather than five offsets being randomly present, the new experiment internixed only three offsets (±2 and 0 sec). Several runs at each separation, with the order counterbalanced, produced a total of about 500–700 trials at each separation. The thresholds were 1.55 ± 0.15, 2.95 ± 0.4, 2.65 ± 0.35, and 1.90 ± 0.15 sec at 1.75, 2.0, 2.25, and 2.5 min, respectively. These data show the same surprising degradation around 2 min; however, with this method, the thresholds are lower (see the section on saturation for a discussion).

The notch at 2-min separation may not be surprising. The observers noted that, for spacings of less than 1.5 min, the lines began to blur together and a luminance cue became available. For example, when the test line had a small downward offset, the eye’s blur function caused the upper gap, between the test line and the upper reference line, to look darker than the lower gap. This luminance cue will be examined in great detail in the sections on control experiments and theoretical results.

Westheimer\textsuperscript{11} reported data for a three-line bisection task for separations ranging from 2 to 4 min, with thresholds between 1/30 and 1/50 of the separation. While at first Westheimer’s results seem to indicate poorer thresholds than those in Fig. 2, the reverse is actually true. In Westheimer’s experiment the test line was an outside line, whereas in the present study the test line was the middle line. If Westheimer’s observers based their judgment on the relative spacing between the middle line and the two outer lines, then their thresholds should be halved in order to compare them with our thresholds. Westheimer’s observers do slightly better than the data shown in Fig. 2, possibly because moving an outside line provides not only the relative spacing cue but also an absolute spacing cue based on the changed distance between the outer lines.

Control Experiments

In order to test whether position judgments for small separations could be explained by a luminance cue, several control experiments were carried out.

Bisection-No Overlap

Simply placing the bisecting line adjacent to, but not overlapping, the reference lines removes the brightness cue. Thresholds were measured for such a stimulus and are shown by the circles in Fig. 2. There are two main points to note: First, for wide separations, thresholds are a factor of 2 higher than for bisection with overlap,\textsuperscript{6} and, second, there is no marked facilitation of thresholds for small separations. Thus, at 1.3 to 1.5 min, the overlap, which would allow a luminance

![Fig. 1](image1.png)

**Fig. 1.** Schematic of the bisection stimulus near the optimal spacing of 1.3 min. The reference lines were presented continuously, and the test line flashed on for 0.6 sec (for subject SK) or 1.0 sec (for subject DL) in one of five positions (0, 1, or 2 pixels above or below the bisection point). For most of the experiments each pixel subtended 1.64 sec at the 10-m testing distance.

![Fig. 2](image2.png)

**Fig. 2.** Bisection thresholds for subject DL as a function of the separation between the lines are shown by the filled triangles. The dashed line has a slope of 1, showing that threshold is a more-or-less constant fraction of the separation down to about 1.2 min. For still closer separations, thresholds increase rapidly. The filled circles show thresholds for bisection with the luminance cue removed by placing the test line adjacent to (see inset) the reference lines (bisection-no overlap). Note that at wide separations these thresholds are a factor of 2 higher than the overlapped condition and there is no sharp decrease in thresholds for small separations.
The reduction in luminance from 0.56 to 0.003 cd/m has only a small (50%) influence on bisection thresholds. Lowering the luminance an additional factor of 2 (to 0.0016 cd/m) resulted in a further factor-of-2 threshold reduction for separations greater than 2 min. Interestingly, bisection was impossible for smaller separations at the lowest luminance. Our interpretation of these data is that, in the Weber region, luminance is relatively unimportant because the judgment is not based on a brightness cue. However, at small separations where the brightness cue is important, the luminance must be large enough to make the cue visible.

Effects of Flanking Lines
When the separation of the reference lines is small, the visibility of the dark line produced by offsetting the test line is hampered by the test line's proximity to the sharp luminance discontinuity that the pattern makes with the dark background. The addition of flanking lines to provide a luminance pedestal similar to the pedestal of the Westheimer effect might be expected to alter the thresholds. Therefore an extra pair of flanking lines was added symmetrically, as shown in Fig. 5. These flanking lines, like the reference lines, were presented continuously. All other experimental details were unchanged. Figure 6 shows how these extra flanks, located at various distances, influence the position thresholds for bisection when the inner separation was 1.3 min (i.e., near the optimal point of the three-line stimulus shown previously). The data for both observers show that flanks at 1.2 min reduce the bisection threshold to around 1 sec. At larger and smaller distances, the flanks produce some inhibition. In fact, for flanks at 1.4 to 1.6 min, the initial thresholds were elevated...
Saturation and the Best Hyperacuity

In all the experiments described thus far, the multicriterion probit analysis assumed a linear transducer (z score linearly proportional to offset). Further analysis of the data (allowing the transducer function to be a power function of the offset with the transducer exponent being a free parameter) showed that the linearity assumption was reasonable for most of the data. However, in the cases when the flanks were optimally positioned, we noticed that the data showed a saturation effect such that the d' (discriminability) between the 0- and 1-pixel offset was greater than the d' between the 1- and 2-pixel offset. If this saturation were real, the thresholds shown thus far with optimally located flanks would be an overestimate. In order to test this, one observer performed a series of experiments without the 2-pixel offset. In this experiment, the stimuli were either 0 or 1 pixel above or below the bisection point. The observer responded up, down, or center. Thresholds

by about a factor of 2 but improved with practice to the level shown. Interestingly, in the optimal region (1.2 min), thresholds were more robust, and little practice was needed. In addition, in this region, blurring owing to less than perfect accommodation seemed to have little effect. This facilitation effect was much stronger for short (4.6-min) lines using the same separations as shown in Fig. 6. With no flanks, the threshold was 3.16 ± 0.31 sec. The addition of flanks improved the threshold by a factor of 2 to a value of 1.63 ± 0.11 sec. With the flanks this stimulus looked like a square approximately 5 min on a side, similar to conditions that give optimal facilitation in the Westheimer paradigm.12

We have repeated these experiments for several inner-separation distances. The data for both observers are shown in Fig. 7. There are several points of note in this figure: (1) For wide inner separations (2 min), the thresholds are independent of the location of the flanking lines. (2) For very small separations, the effect of the flanking lines is dramatic. For example, for the 0.8-min inner separation, the three-line target (flanks at +) has a very high threshold (>6 sec). The extra flanks strongly facilitate position discrimination, lowering threshold by about a factor of 3 in the optimal location. (3) The optimal separation of the outer flanks tends to be slightly less than the inner separation. This finding is interesting since it provides a critical test of any theoretical analysis. Moreover, the finding that flanks have no influence on thresholds for separations > 2 min provides further evidence that separate cues are used for bisection of large and small spaces.

Fig. 6. Bisection thresholds (arc sec) for SK (open circles) and DL (filled circles) as a function of the distance of the flanking lines. The inner separation was 1.3 min. The data at the far right-hand side are for the three-bar stimulus where flanks are not present.
calculated from eight runs of the same condition (outer separation 1.24 min, inner separation 1.3 min) are shown in Fig. 8. These results show that the position threshold was always below 1 sec, with the average and its standard error being 0.85 ± 0.04 sec. The 0.85-arc-sec displacement that was visible on 75% of the trials amounts to a linear displacement on the retina of 0.07 μm, which is one eighth the wavelength of yellow light! This is satisfying because it improves on the present world record for visual discrimination held by a German student who purportedly can identify people at a distance of 1.6 km. However, our best position threshold is quite similar to the minimum width of a visible black line, providing further support for a luminance cue.

SPACE-SIZE ANALYSIS

In many tasks that require fine spatial discriminations the observer attends to a particular localized feature of the pattern. In these circumstances it is useful to analyze the stimulus based on the output of a continuum of filters of varying size (spatial frequency) and location. We shall call the output of the space–size analysis a space–size plot or viewprint, in analogy with the voiceprint used in audition. A viewprint of a pattern is equivalent to viewing the pattern through an array of bandpass spatial filters of different sizes and positions such as illustrated in Fig. 9. The purpose of having both even- and odd-symmetric mechanisms, as illustrated in Fig. 9, is discussed below. Since the filters are localized in both space and spatial frequency, the viewprint is a representation of the image that simultaneously conveys the spatial and the spatial-frequency structure of the pattern. Viewprint analysis is a modification of the spatial-filtering approach, which looks at a scene through a small number of filter sizes at well-separated spatial frequencies. The viewprint approach, however, stresses the continuous nature of size sampling (see two bottom panels of Fig. 13 below).

There are many advantages of the dual space–size approach:

1. There are features that appear in the viewprint that do not stand out either in the original spatial profile of the stimulus or in its Fourier transform. It will be shown that viewprints of our bisection stimuli are in good agreement with the subtle nuances of our bisection data. Our observers seem to be attending to localized features of the viewprint.

2. Modern electrophysiology and psychophysics have shown that the primate visual system contains mechanisms (receptive fields) at a variety of spatial locations and with a variety of receptive-field sizes, shapes, and symmetries. It is plausible that the human visual system uses these mechanisms to make position and size judgments.

3. Absolute position is poorly encoded by the visual system because of eye movements and because of an inaccurate sense of direction of gaze (corollary discharge). Viewprint analysis will be shown to be robust to the presence of eye movements. This is of particular importance in accounting for hyperacuities, in view of their relative insensitivity to retinal image motion.

4. As was emphasized by Westheimer, standard Fourier analysis has limited applicability to spatially localized patterns. Furthermore, there is little evidence that the visual system is organized according to a Fourier scheme, whereas it does have obvious retinotopic spatial organization. Viewprints allow local Fourier analysis to be applied to the stimuli. Appendix A shows the power of the frequency approach in setting the limits to hyperacuity.

Overview

The computation of the viewprints requires several steps, which are illustrated in Fig. 10 and discussed in detail below. These steps are as follows: (1) A new class of receptive fields (the Cauchy functions) is introduced (Fig. 10A). These functions have particular elegance for visual modeling. (2) The receptive-field's sensitivity is normalized by the contrast-sensitivity function, as shown in Fig. 10B. (3) A contrast-response function, shown in Fig. 10C, is developed for extending the formalism into the suprathreshold regime. The response is normalized to the signal detection threshold. The contrast just-noticeable difference (JND) contrast change required to produce Δd′ = 1 is shown in Fig. 10D. The dip in this function implies that there is a range of background contrasts for which contrast discrimination is easier than contrast detection. The effective contrast (dotted curve in Fig. 10D) is the inverse of the Weber fraction and is discussed in Section 4 of Appendix A. (4) The Pythagorean sum of even- and odd-symmetric filters produces a response that is independent of absolute phase. (5) A scheme (viewprint) for graphically presenting much of the information is developed, and one bit of phase information (the sign) is preserved. (6) The differential response (response to a signal minus the response to a blank) is computed and displayed. The differential response indicates the sizes and locations of mechanisms that are maximally sensitive to the discrimination cue.

Receptive-Field Shapes

The receptive-field shapes that we use are based on functions described by Cauchy, which have spatial-frequency tuning given by

$$C_n(f) = f^n \exp(-f).$$

(1)

This function is also the Poisson probability distribution. To avoid confusion with Poisson statistics (concerned with the distribution of n rather than of f), the filter functions will be referred to as Cauchy functions. This set of functions will be shown to have many advantages for modeling the visual system.

The Cauchy function C_n(f) has a peak at f = n. Since the
integral of $C_n(f)$ from 0 to $\infty$ equals $n!$, it is easy to calculate moments of $C_n(f)$ on a linear frequency scale. The mean is $f = n + 1$, and the variance is $n + 1$. In order to vary the peak spatial frequency of the filter, a scaling parameter $\sigma$ is introduced so that the spatial-frequency tuning is given by $C_n(\sigma f)$.

The peak of the function now occurs at $n/\sigma$. Thus the mechanism with $\sigma = 2$ min and $n = 4$ has a peak at $f \approx 2 \text{rad/min} \approx 20 \text{cycles per degree (c/deg)}$. The symmetric (S) and antisymmetric (A) spatial receptive fields [proportional to the Fourier transforms of $C_n(\sigma f)$] are

$$S_{n-1}(t) = \Re(1 + it)^{-n} = \cos^n(\theta)\cos(n\theta),$$

$$A_{n-1}(t) = -\Im(1 + it)^{-n} = \cos^n(\theta)\sin(n\theta),$$

where $t = \tan(\theta) = x/\sigma$. The higher order receptive fields are derivatives of lower-order receptive fields:

$$S_n = \sigma/\!\!dA_{n-1}/dx,$$

$$A_n = -\sigma/\!\!dS_{n-1}/dx.$$
plotted in Fig. 11 to illustrate how they differ from one another.

Table 1 shows two separate measures of symmetry for the frequency tuning: the half-bandwidths and the half-standard deviations. The half-bandwidth is the distance in octaves needed for the sensitivity to fall to half of the peak value. The half-standard deviation can be conceptualized by splitting the tuning curve in half at the peak and then creating the mirror image of the upper or lower half. Both measures show the Cauchy functions to be the most symmetric functions (on an octave scale), compatible with psychophysical experiments using adaptation and masking. Table 1 shows that the Cauchy and Gabor functions, respectively. However, psychophysical experiments using adaptation and masking (see also the discussion in Ref. 27) reveal underlying mechanisms that have a sharper low-frequency falloff (a ratio of slightly less than unity). The Gabor and Gaussian filters have a strong asymmetry in the opposite direction. Thus the Cauchy functions provide a closer match to the psychophysical results.

Note that in Table 1 both the symmetric and the antisymmetric Gabor functions have an infinite standard deviation on the low-frequency side. This occurs because the low-frequency falloff is very gradual on an octave scale. The symmetric Gabor functions have a nonvanishing zero-frequency response, whereas the antisymmetric Gabor functions have a linear falloff near zero frequency. The Cauchy functions C3 and C5, on the other hand, have much steeper falloffs, decreasing with the third and the fifth powers of frequency. On a linear axis the Gabor functions have the desirable property of minimizing the product of the spatial and spatial-frequency standard deviations. However, on a logarithmic (octave) frequency scale, the symmetric Gabor functions maximize the uncertainty. A logarithmic axis is appropriate since a multitude of psychophysical results show similar mechanism tuning curves on a logarithmic axis. A logarithmic axis is used in all our viewprints, and it is desirable to minimize the area covered by each mechanism on these coordinates.

**Size of Inhibitory Flanks**

The receptive-field properties in Table 1 show differences between these broadly tuned mechanisms. The Gabor function is a product of a cosine (or sine) and a Gaussian and therefore has evenly spaced zero crossings. The Cauchy functions, on the other hand, have inhibitory flanks that are wider than the center width, consistent with psychophysical studies and in disagreement with the Gabor functions.

**Calculation of Sensitivity**

Since we are attempting to predict the absolute magnitude of the hyperacuity threshold, the normalization factors must be carefully considered:

**Normalization Hypothesis**

An upper limit to the mechanism sensitivity is set by the CSF. The mechanisms used for the hyperacuity judgment cannot be too sensitive, or they could detect gratings with lower contrast than specified by the CSF. On the other hand, the normalization of the hyperacuity mechanisms could be lower.

![Fig. 11. Upper panel: The symmetric receptive fields whose characteristics are given in Table 1 are plotted. Curves C3 and C5 are Cauchy functions; G4 and G6 are Gabor function whose variances are 1/4 and 1/6; G* is the second derivative of a Gaussian. The tic marks indicate σ = ± 1 for the Cauchy distribution. Middle panel: Antisymmetric receptive fields for the Cauchy and Gabor functions. Lower panel: Spatial-frequency profiles of the three filter types. Two profiles are drawn for the Gabor functions since the symmetric and antisymmetric filters have different tuning. The horizontal axis is on an octave scale, with the tick marks indicating octave intervals. In all plots the vertical axis is linear.](image_URL)
(less sensitive) than specified by the CSF, corresponding to a poor efficiency of the hyperacuity judgment. We will assume, however, that the hyperacuity mechanisms have approximately the same sensitivity as the grating mechanisms. Our results will be found to be consistent with this assumption.

Shape of the Contrast-Sensitivity Function

A reasonable estimate of the CSF is based on its linear behavior (log sensitivity versus linear spatial frequency) at high spatial frequencies where the CSF decreases by about 1 dB per c/deg. The magnitude of the CSF is such that a 56-c/deg grating has a threshold at 200% contrast (thin bright lines on a dark background—similar to our stimuli), a 50-c/deg grating has a threshold at 100% contrast (in agreement with the cutoff spatial frequency of our observers measured with square-wave gratings), a 30-c/deg grating has a threshold at 10% contrast (20 dB lower), and a 10-c/deg grating has a threshold at 1% contrast. At low frequencies the CSF begins to decline. A functional form for the CSF that provides a close match to the CSF data is given by

\[ \text{CSF}(f) = A(f\rho)^m \exp(-f\rho), \]  

where \( A = 600, m = 0.5, \) and \( \rho = 1.25 \text{ min/rad}. \) This function has a peak at \( f = m/\rho = 4 \text{ c/deg}. \) Note that the CSF has exactly the same form (a Cauchy function) as the individual mechanisms, except with different parameter values for \( m \) and \( \rho. \) Kelly has used a similar Cauchy function to fit the CSF, except that his value of \( m \) is between 1 and 2. The value of \( m \) is strongly influenced by the temporal aspects of the stimulus presentation. Since our experiments were done under free viewing conditions with eye movements allowed, \( m \) is small. The precise value of \( m \) does not concern us since \( m \) governs only the low-spatial-frequency characteristics of the CSF and has negligible effect on our modeling of the hyperacuity task.

Multiple-Channel Envelope

There are two methods by which the underlying mechanisms can generate the CSF: (1) The CSF can act as an initial filter, which differentially attenuates different frequency components of the stimulus. This would be the case if the CSF were entirely due to optical factors, such as diffraction and spherical aberration. In this case the stimulation of a mechanism is

\[ S(\sigma, f) = \text{CSF}(f)C_n(\sigma f). \]  

Since the product of two Cauchy functions is another Cauchy function (of order \( n + m \)), the product maintains a constant bandwidth and receptive-field shape independent of \( \sigma. \) For mechanisms with non-Cauchy frequency tuning, the CSF filter distorts the receptive-field shape. (2) The CSF can be generated by mechanisms having different sensitivities \( N(\sigma) \) rather than by an initial filtering, so that the mechanism stimulation would be given by

\[ S(\sigma, f) = N(\sigma)C_n(\sigma f). \]  

The present calculations take the latter approach, since below 50 c/deg the CSF is limited primarily by neural blurring rather than by optical blurring. The CSF as generated by the envelope of the Cauchy functions is shown in Fig. 10B for \( n = 3 \) and \( n = 5. \)

By taking both the CSF tuning and the underlying mechanism tuning to be Cauchy functions it is irrelevant whether the CSF is produced by method (1) or by method (2). Any function other than a Cauchy function requires an assumption about the relative contributions of optical factors and neural factors in determining the CSF. For broad-bandwidth filters the difference between these two assumptions can be large.

In our calculations the mechanism sensitivity is normalized by first finding the frequency at which the mechanism is tangent to the CSF. This is the point at which that particular mechanism is the most sensitive mechanism for detecting a grating. The tangent frequency for Cauchy functions is given by the elegant simple formula \( f_t = (n - m)/(\sigma - \rho), \) where \( n \) and \( \sigma \) specify the Cauchy function for the mechanism and \( m \) and \( \rho \) specify the Cauchy function for the CSF [Eq. (4)]. For a flat CSF \( (m = 0 \text{ and } \rho = 0), \) the tangent frequency is equal to the peak frequency of the mechanism, as expected. Once the tangent frequency \( f_t \) is known, the normalization \( N(\sigma) \) is straightforward to calculate from Eq. (6), which gives

\[ \text{CSF}(f_t) = N(\sigma)C_n(\sigma f_t). \]  

Probability Summation

Before the CSF can be used to set the sensitivity of an individual mechanism, probability summation must be taken into account. The CSF is determined by the grating-detection threshold, which is based on the pooled activity of many mechanisms at different positions. Several authors have shown that grating sensitivity increases as a function of the number of cycles. For the hyperacuity task minimal spatial probability summation is expected, since the observer attends to particular localized features of the pattern. At the high spatial frequencies of interest to us we assume that the region of probability summation for grating detection is limited to the central fovea, so only about the central 15 cycles are involved. The effect of probability summation in going from 1 to 15 cycles is about a factor of 2, so we assume that the underlying mechanisms effectively have half of the sensitivity that would have been expected from a localized group of mechanisms. This factor does not really give an individual mechanism’s sensitivity, since that quantity would require knowledge of the sampling grain of the cortex. The factor of 2 is merely the relative factor that adjusts for attention to many cycles in the CSF measurement versus attention to a single cycle in the hyperacuity judgment. The CSF shown in Fig. 10B is normalized by \( A = 600 \) in Eq. (4). However, the CSF that we use to obtain the mechanism sensitivity in the viewprints uses a normalization of \( A = 300 \) to compensate for the probability summation across space that is present for grating detection. A compensation factor is not needed for probability summation across spatial frequency since an essential aspect of our viewpoint analysis is that the hyperacuity judgment is based on a range of spatial frequencies that is narrow with respect to the bandwidth of the mechanisms.

The term probability summation may be inappropriate for visual modeling, since it assumes that the decision stage is based solely on the mechanism responding maximally to the signal plus noise. A more reasonable decision stage would give some weight to nearby mechanisms whose response is slightly less than maximal. Thus the neutral term “pooling” may provide a better description of the extent to which several
mechanisms contribute to the discrimination decision. The extent of pooling in spatial frequency may be very limited (about 0.125 octave). The extent of pooling in spatial position, on the other hand, may be quite significant, at least on the hyperacuity scale. We shall assume that the decision stage spatially pools the output of mechanisms over a range whose standard deviation is at least 0.5 min in extent. Surprisingly, fine hyperacuity judgments are predicted even though absolute position information is degraded by spatial pooling. The extent of pooling will be discussed further in the section on the differential response and in the discussion section.

Local Contrast
The normalization conditions considered in this section have been developed for the spatial-frequency domain. But our bisection stimuli were localized in space rather than in spatial frequency. Therefore, in calculating the stimulation of each mechanism, we use the spatial receptive fields as defined by the Fourier transform of Eq. (6). The major difficulty in going to the space domain is the problem of defining the local contrast of the stimulus. If we simply treated the mechanisms as linear filters, then doubling the intensity of the lines would double the stimulation and would double the visibility of the bisection offset. However, as is shown in Fig. 4, the bisection thresholds are almost independent of luminance. Thus, before convolving the mechanisms with the stimulus, the stimulus must be normalized to contrast units. We used the following pragmatic strategy for normalization. The receptive field, as defined by Eqs. (2) or (3), was applied to a grating composed of 25 equally spaced lines. We examined the stimulation of those mechanisms whose location was near the center of the grating and whose peak spatial frequency was more than an octave below the stimulus spatial frequency. The reason for choosing the center was to ensure that the full receptive field was being stimulated. The reason for choosing to examine low-spatial-frequency mechanisms was to ensure that the second and higher harmonics of the stimulus gave a negligible contribution. Under these conditions the stimulus was indistinguishable from a 200% contrast sinusoid. A simple analytic normalization factor was found by which the spatial analysis gave the same amount of stimulation as the frequency analysis shown in Fig. 10B for any spacing of the grating. This normalization factor was used throughout our calculations. Further consideration of this normalization factor is taken up in the discussion section.

Contrast-Response Function
Since the line stimuli used in our (and indeed in most) hyperacuity experiments were quite visible, it is important to develop a means to compute the response of mechanisms to suprathreshold contrast levels. In order to calculate the visibility of a stimulus, the signal-to-noise ratio of the underlying mechanisms in response to that stimulus must be known. In our analysis the signal-to-noise ratio is expressed in units of the signal detection parameter $d'$. A $d'$ value of 2 means that the signal is 2 standard deviations above the noise. The function relating $d'$ to the stimulus strength is called the transducer function or the contrast-response function.

Our basic hypothesis is that the mechanisms involved in hyperacuity tasks have the same sensitivity as the mechanisms that are involved in contrast detection and contrast discrimination. Thus the contrast-response function derived from the contrast domain is assumed to be applicable to other visual judgments. The contrast-response function used in our calculations is

$$d' = \ln(1 + S/W)/\ln(1 + 1/W),$$

(8)

where $S$ is the amount of stimulation of an individual mechanism and $d'$ is the signal-to-noise ratio of that mechanism. This contrast-response function incorporates a number of properties that are in agreement with psychophysical experiments: (1) A threshold stimulus $(S = 1)$ is normalized (by the denominator factor) to give $d' = 1$. The actual predictions, however, will be for $d' = 0.675$, corresponding to 75% correct. (2) A choice for the transducer exponent of $r = 2$ in agreement with a wide variety of detection tasks, which show that $d'$ is a quadratic function of contrast for low contrasts. (3) At higher contrast levels the contrast-response function is highly compressive. The parameter $W$ governs the point at which $d'$ begins to saturate. (4) We determine the value of $W$ by the contrast-discrimination jnd. Contrast discrimination is specified by the change in stimulus ($\Delta S$) that is required to produce a unity change in $d'$. For $r = 2$, Eq. (8) can be solved exactly for $\Delta S$, giving $\Delta S = [(1 + 1/W)S^2 + 1]^{1/2} - S$. This curve is plotted in Fig. 10D. For small $S$ this function is a parabolic dipper function, and at large $S$ it is proportional to $S$ with a Weber fraction of $\Delta S/S = (1 + 1/W)^{1/2} - 1$. For the present calculations, a saturation point of $W = 2$ is assumed, corresponding to a Weber fraction $(\Delta S/S)$ of 0.22. The 20% value for contrast discrimination is similar to the contrast jnd for local stimuli found by Legge and Kersten and Badcock. Section 3 of Appendix A shows how a constant-contrast Weber fraction can be related to the observed “Weber” fraction for size judgments. The reciprocal of the Weber fraction $(S/\Delta S)$, which is plotted in Fig. 10D, can be considered the effective contrast of the stimulus for discrimination experiments. For low contrasts the effective contrast is equal to the contrast. Near threshold the accelerated transducer function causes the effective contrast to be greater than the real contrast. Above the saturation point it levels out to a value of $1/(1.5 - 1) = 1.45$ times the threshold contrast. Section 4 of Appendix A discusses effective contrast and shows how the contrast Weber fraction can be related to the threshold for detecting a displacement of a sinusoidal grating.

Our choice of a logarithmic contrast-response function might be questionable since a power law such as $\Delta S \approx S^{0.6}$ is commonly found. Thus a more accurate parameterization of the contrast-response function might be to replace the logarithms in Eq. (8) with the power function $(1 + S/W) = 1/p$, where $p = 0.2$ leads to the power law $\Delta S \approx S^{0.6}$. However, a small power such as $p = 0.2$ is not much different from the logarithm in Eq. (8) since the power function equals a logarithm in the limit as $p$ approaches zero. In addition, at this early stage in the development of models of spatial vision, the accuracy of the models does not warrant the introduction of an extra parameter, such as $p$.

The contrast-response function given by Eq. (8) is strongly compressive at high stimulus strengths. It produces $d'$ values of 2.7, 10, and 21 for stimuli that are 2, 10, and 100 times threshold. Other signal-detection assumptions can produce contrast-response functions with much less compression. For example, Nachmias and Kocher find that $d' = 10$ when the
stimulus is only 2.5 times threshold. The difference between the two signal-detection procedures is that Nachmias and Kocher define $d'$ to be the horizontal intercept of the receiver-operating-characteristics (ROC) curve, whereas we define $d'$ to be the vertical intercept. The horizontal intercept uses the standard deviation of the noise distribution as the measuring stick, and the vertical intercept uses the signal distribution as the measuring stick. Suprathreshold experiments measure the discriminability between closely spaced contrasts. The contrast jnd is in fact the $d'$ measured in units of the signal distribution—the vertical $d'$. The horizontal $d'$ is unreliable since it requires knowledge of the size of the signal standard deviation in terms of the noise standard deviation—a poorly known quantity. Near threshold, the $d'$ defined by Eq. (8) maintains its usual signal-detection meaning. At suprathreshold levels, however, our definition of $d'$ becomes a measure of the number of jnd's above threshold. A viewprint value of 12 means that the mechanism is being stimulated 12 jnd's above threshold.

**Pythagorean Sum and Eye Movements**

In normal vision the eyes are constantly moving. It is thus not surprising that the visual system is poor at judging absolute position. It is much better at judging position relative to a nearby reference. Figure 2 amply demonstrates how the position judgment is degraded as the reference lines are separated. One of the goals of our space-size analysis is to develop a formalism that maintains high-quality relative-position information while degrading absolute-position information. Such a scheme would have the advantage of being tolerant to image motion, in agreement with several hyperacuity experiments that have compared moving versus stationary targets.

The crux of our analysis is to preserve full information about the magnitude of stimulation while discarding phase information. The magnitude of stimulation is calculated by the Pythagorean sum of the symmetric (SS) and the antisymmetric (SA) stimulation. The contrast-response function [Eq. (8)] thus becomes

$$d' = 2.5 \ln[1 + (SS^2 + SA^2)/2].$$

The Pythagorean sum may be a rather general strategy in visual processing to eliminate confusion caused by rapidly varying phase. For example, Adelson and Bergen also use it for modeling motion perception. It is unlikely that the visual system actually carries out a Pythagorean sum per se. Rather, a quasi-Pythagorean sum may be obtained by pooling the outputs of adjacent mechanisms with a variety of phase sensitivities. Any scheme in which the response is based on a pool of equally sensitive mechanisms with a continuous range of phases is equivalent to the Pythagorean sum (see Appendix A of Ref. 15 for alternative Pythagorean sum calculations).

One of the interesting features of the Pythagorean sum is that it places a reduced burden on higher stages of processing. If, for example, the visual image were encoded by the location of the zero crossings of the symmetric mechanism, then the precise location of the zero crossing would have to be stored with high accuracy. A zero-crossing scheme places greater emphasis on absolute-position information than is exhibited by the visual system. The viewprint scheme that we are developing, on the other hand, preserves just the information that seems to be preserved by the visual system.

**Summary of Viewprint Calculation**

The viewprint calculation involves the following steps:

1. Normalize each of the filters in Fig. 9 by $N(\sigma)$ from Eq. (7).
2. Calculate the overlap of each filter with the stimulus expressed in contrast units.
3. Calculate $d'$ using the contrast-response function [Eq. (9)].
4. Plot the results (see the next section).

**VIEWPRINT RESULTS**

Viewprints

Voiceprints are commonly used to recognize auditory patterns, such as speech and bird songs. For application to visual stimuli several modifications to the classical voiceprint have been made and are illustrated in Fig. 12, which shows two viewprints, based on the C3 mechanisms, of a three-line binocular stimulus for a separation of 4.0 min. The test-line offsets in the upper and lower panels are 0.04 and 0.08 min.

![Fig. 12. Viewprints for three-line stimuli for the C3 mechanism. The interline separation was 4.0 arc min. The two panels correspond to two offsets of the test line. Lower panel: Threshold offset (0.08 min, corresponding to a 1/50 Weber fraction). Upper panel: A half-threshold (0.04-min) offset. The shading represents one bit of absolute phase (the sign). The contour lines are the iso-$d'$ lines of the contrast-response function as indicated by the numbers by the lines. The clusters of slightly smaller numbers are the differential response and indicate the discriminability of the offset. Negative differential-response values are indicated by the reversed contrast numbers.](image-url)
where the threshold offset is 0.067 min. Several characteristic features of our viewprints are pointed out as follows:

1. The curved lines throughout the viewprints are iso-$d'$ contour lines. A disadvantage of the contour plots is that without training it is difficult to get an intuitive grasp of the terrain. A clarifying perspective plot of the contours will be shown and discussed below in connection with Fig. 13 (two bottom panels).

2. The $d'$ values (1, 6, 10, 14, and 18) are shown and indicate the value of the contrast-response function ($d'$) at the corresponding contour line.

3. To help to interpret Fig. 12, dotted lines show contour lines that surround a depression.

Fig. 13. Viewprints for three-line stimuli with different separations for the C3 and C5 mechanisms. Each horizontal grouping of two viewprints is for a different three-line stimulus with interline distances of 1.0, 1.2, 1.5, and 2.5 min. The test lines were offset by the threshold levels of 0.05, 0.026, 0.03, and 0.05-min. Other details are the same as in Fig. 12. The bottom panel is a perspective view of the $d'$ response to the three-line stimulus with 2.5-min separations and 3-sec offset. The dashed lines can be used for calibration since their horizontal positions coincides with the stimuli, and the vertical extent of each dash corresponds to $d' = 2$. 
(4) The shaded regions convey one-bit phase information (the sign). A problem with the classical voiceprint is that phase information is lost. That is, the activity of a bright-bar mechanism and the activity of a dark-bar mechanism are indistinguishable. This works well for audition, in which a waveform reversed in sign sounds the same as the original. However, in vision there is psychophysical and anatomical evidence for separate processing by on-center and off-center mechanisms (see the discussion section). In order to preserve some absolute phase information in the viewprints, one bit of the phase is displayed. This is done by shading those regions of the viewprint wherever the output of the symmetric mechanism is negative. The unshaded regions are where bright-bar mechanisms are stimulated. The shaded regions are where dark-bar mechanisms are stimulated. This scheme also enables bars and edges to be differentiated even in the presence of eye movements smearing the location of the boundary that defines one-bit phase.

(5) The boundary between the shaded and unshaded regions are the locations of the zero crossings that have been useful in computer vision.16

Figure 12 shows that the $d'$ contours and the zero-crossing contours are decoupled. These two types of contour might be processed separately by the visual system, in analogy to the finding of Treisman40 that form (similar to the $d'$ contours) and color (similar to spatial phase) are not “glued” together but rather are separable dimensions. Our viewprint representation provides a way of separating these two aspects.

(6) A dot is located at the bottom of each shaded region and marks the location of a null point. The dot occurs at the bottom of each shaded region because that is the point at which the stimulation of both the symmetric and antisymmetric mechanisms changes sign.

(7) Possibly the most important feature of the viewprints are the numbers, which are found in several clusters. The numbers in these active regions are the change in $d'$ produced by offsetting the middle test line. These numbers represent the differential response. The differential-response values that are shown on the viewprints are calculated as follows: (1) The $d'$ values from the contrast-response function are spatially blurred by convolving them with a Gaussian blur function whose standard deviation is 0.5 min. The $d'$ values have been smeared in accord with the degradation of local spatial information by spatial sampling. The 0.5-min spatial blur corresponds to the hypothesis that a feature can not be localized more accurately than about one fourth of a hypercolumn. In peripheral vision and in strabismic amblyopia we expect that a broader blur function may be necessary. It might also be true that a broader blur function should be used for the large mechanisms since the decision stage may not localize the larger mechanisms as accurately as it does the tiny mechanisms. For simplicity, however, we have used a constant 0.5-min blur throughout. It is always possible to blur the differential response further, based on the output shown in Figs. 12–14. (2) The $d'$ for an undispaced test line (also blurred) is subtracted from the $d'$ for the displaced test line. (3) This $d'$ difference is divided by 0.675 since in our experiments the threshold criterion was chosen to be $d' = 0.675$ (75% correct) rather than $d' = 1$. (4) The final step is to round off and truncate the differential response so that it can be plotted as an integer.41 This procedure results in a differential response, which is expressed as an integral number of threshold units. Because of the rounding algorithm, differential responses that are between 0.75 and 1.75 times threshold are shown by the numeral 1 in the viewprints. The region with positive differential responses is in general larger than the region with negative differential responses, because there is less stimulation of the mechanisms when the test line is at the bisection point. If the magnitude of the differential response is greater than unity, an ideal observer should be able to detect the change. In the discussion (the section on response strategies) we speculate on how the differential response might be used by the visual system. We do not propose that the visual system actually takes the difference between the response to the test pattern and the memorized response to a standard pattern.

A model based on the differential response of a localized region of the space-size plot had been proposed earlier27 to account for the facilitated detection of a third harmonic in the presence of a fundamental. It was shown that a small group of mechanisms located near the zero crossings of the fundamental and whose spatial-frequency tuning curve peaked near the second harmonic had a differential response that matched the data. A similar model based on localized cues seems able to account for phase discrimination of compound gratings.

Cues for Bisection-Size Judgments

A multiplicity of cues can be used in the bisection task whose viewprint is shown in Fig. 12. The cues will be discussed according to their position along the spatial-frequency axis of the viewprints (from top to bottom).

High Spatial Frequency

The two nonzero differential response values toward the top of Fig. 12B for the 0.08-min offset correspond to a cue for the absolute position of the line. Tiny mechanisms are sensitive to the absolute location of the test line. The active region is not directly over the line location but rather at the location where the contrast-response function changes most rapidly. This result is quite different from a model in which the observer could attend to a mechanism with precisely specified location and phase. For the latter model the optimal strategy would be to attend to an edge mechanism centered on the test line. There are several problems with using the absolute line location for making size (bisection) judgments: (1) Constant eye movements make it difficult to relate the absolute position of the line to the moving reference lines. (2) The spatial-frequency shift following grating adaptation43 would be hard to explain. The same arguments militate against the use of zero crossings (one-bit phase) for size cues; moreover, as Watt43 has pointed out, zero crossings are not robust to noise. We hypothesize that the visual system largely ignores cues based on absolute position. Furthermore, as is seen in Fig. 12B, the high-frequency position cues are not so strong as the cues at lower frequencies, which are discussed next.

Although the high-frequency cues will not be considered further, they should not be dismissed since they provide cues for position information but with higher thresholds. Viewprint calculations for the detectability of a shift of the entire pattern show that, in order for the differential response of the high-frequency mechanisms to be above threshold, the shift must be greater than 0.2 min. At these higher shift values the observer may also detect shifts in absolute phase, as is discussed in Section 4 of Appendix A.
Middle Spatial Frequency

By middle spatial frequency we mean spatial frequencies whose period is about the same size as the interline separation of the stimulus. For the stimulus shown in Fig. 12 the spacing of 4 min is associated with a fundamental spatial frequency of 15 c/deg. Because of the rapidly decreasing CSF, the 15-c/deg stimulus will be detected by mechanisms peaking at 10.1 and 11.6 c/deg for the \( n = 3 \) and \( n = 5 \) Cauchy mechanisms. Middle spatial frequencies for this stimulus can be considered to extend down to about 10 c/deg. One influential model for size judgments proposed by Blakemore and Sutton\(^4\) is based on the most actively firing mechanism. The difficulty with this model is that a small change in the stimulus spacing produces only minor changes in the stimulation of the peak firing mechanisms. The maximal change in stimulation occurs where the spatial-frequency tuning function is changing most rapidly. An ideal observer would thus place greater weight on mechanisms sensitive to spatial frequencies above and below the stimulus spatial frequency. The lack of any differential response between 15 and 9 c/deg in Fig. 12 makes it unlikely that these cues are useful in the present task.

Another cue in the middle-spatial-frequency region is the total area of each shaded region corresponding to the one-bit phase cue. This cue measures the visibility of the dark bar and may be related to the mass measure of Watt\(^3\) and also to the local-contrast measure of Badcock.\(^7\) This use of phase information is quite different from using the absolute location of the zero crossings since it is still useful in the presence of

Fig. 14. Viewprints for five-line stimuli. For each viewprint, the inner line separation is 1.3 min and the offset is 0.02 min. The separations between flanking and reference lines (see Fig. 5 for nomenclature) are 1.0, 1.2, 1.5, and 2.0 min and \( = \) (a three-line stimulus). Other details as in Fig. 12. The presence of the flanks is seen to increase the differential response of the viewprints, in agreement with the experimental results shown in Fig. 6.
eye movements. The area cue also has appeal because it corresponds to the cue that the subject believes he is using for closely spaced lines. The precise relationship between Badcock’s local contrast and our \( d' \) measure remains to be determined.

Low Spatial Frequencies

Regan and Beverley\(^4\) recently reported a stunning finding that prior adaptation to a grating interferes with size discrimination of a test grating if the adapting frequency is about an octave below the test frequency. This would imply that low-frequency mechanisms are used for size judgments. In order to account for their data, Regan and Beverley assumed that the underlying mechanisms have a much steeper falloff on the high-frequency side than on the low side. Psychophysical evidence based on masking\(^2\) and adaptation\(^3\) however, reveals the opposite asymmetry, that the steeper falloff is on the low-frequency side. The steepness of the frequency tuning is critical for the size-discrimination models of Regan and Beverley\(^4\) and Wilson and Gelb.\(^4, 40\) Section 5 of Appendix A shows that models based on the steepness of the frequency-tuning curve might be able to produce frequency discrimination using sinusoidal gratings can be achieved using other mechanisms and cues. (See the discussion in Sections 3 and 5 of Appendix A for a simplified analysis showing how a 1-part-in-30 bisection threshold can be related either to the contrast–discrimination function or to the steep slope of the frequency-tuning function.)

The viewprint in Fig. 12B for the 4-min separation and 0.08-min offset (1.2 times threshold) with the C3 filter has several differential response values that are greater than unity, implying that this stimulus offset should be discriminable, whereas the 0.04-min offset shows only a few unity differential responses (in part because of the rounding up of the calculated integer value). The differential response values shown in Fig. 12 (similar results can be seen in Fig. 13) indicate that for widely separated lines the model predicts thresholds that are slightly better than those observed experimentally (notice several values equal to 2 in Fig. 12B). One possible explanation for this discrepancy is that for low-spatial-frequency mechanisms the response pooling may have an extent larger than the 0.5-min standard deviation that was assumed. A larger pooling extent would provide further dilution of the position cue.

Weighted Average

An alternative model based on a weighted average of the filters in which the spatial frequency of each mechanism is weighted by its response. The linear weighting is similar to the strategy of an ideal detector, which uses the output of filters to assess the spatial frequency of a single sinusoid. To the extent that the linear weighting emphasizes the mechanisms near the tails (especially the low-frequency tail) of the tuning curve, this model may be compatible with the cues discussed above. Further study is needed to determine the relationship of this model to the others.

Viewprints for the Bisection Experiments

Figure 13 shows viewprints for the three-line bisection stimulus for separations of 1, 1.2, 1.5, and 2.5 min. The test lines were offset by 0.05, 0.026, 0.03, and 0.05 min, corresponding to the empirical threshold levels. The left- and right-hand groups of viewprints are for \( n = 3 \) and \( n = 5 \), corresponding to the Cauchy filters C3 and C5. In each case the region near the null points has a differential response of about unity. Thus an ideal observer limited by the human contrast-response function would have about the same hyperacuity threshold as was found by our human observers. Closer inspection reveals that the C3 filter is not quite sensitive enough (by about 20%) to account for the bisection threshold for the 1.2-min separation, and the C5 filter is too sensitive to account for thresholds for separations greater than 2 min. This finding is compatible with results from other studies, which show that the higher-spatial-frequency mechanisms have slightly narrower bandwidths.\(^2, 25\)

The two bottom panels of Fig. 13 show perspective views of the contour plots for the 2.5-min separation. The purpose
of the perspective view is to aid the interpretation of the features depicted by the contours. The vertical axis is the $d'$ response of a filter. Each curve is for a single filter size. The separation in filter size between adjacent curves is 0.0625 octave (half of the separation used for calculating the viewprints) with coarser filters at the bottom. The profiles at 0.5-octave spacing (21.21, 15, 10.61, and 7.5 c/deg) are indicated. The horizontal axis specifies spatial position from -6 to +6 min. The vertical dashed lines correspond to the locations of the three stimulus lines. The height of each dash corresponds to $d' = 2$, and the vertical axis for each curve is linear in $d'$. The height of each curve can be determined by carefully tracing the curve from the edge of the figure where the curve meets the x axis.

The perspective profiles of $d'$ show several interesting features, which may not be so obvious from the viewprints. The C3 mechanism responds individually to each of the three lines of the stimulus. The C5 mechanism, however, shows either a flat response across space or a double peak near the outside two lines. This feature shows the trade-off between space and frequency. The broader-bandwidth mechanisms (C3) are more localized spatially and are thus able to respond to the individual lines. The central dip found for C5 is interesting and shows the effectiveness of the inhibitory side-lobes in suppressing the response to mismatched spatial frequencies.

The most prominent features of the perspective plots are the sharp dips where the $d'$ approaches zero. The zeros occur at isolated points of the space-size plot where neither symmetric nor antisymmetric mechanisms are being stimulated. These points, which occur at about half of the spatial frequency of the line stimuli, are the most sensitive mechanisms for hyperacuity discriminations.

It is useful to divide the viewprints into two groups, depending on whether the null points fall in a low $d'$ region of the contrast-response function ($d' < 2$) or in a high $d'$ region. The top six viewprints of Fig. 13 fall into the first class. The two bottom viewprints (a 2.5-min separation) and the viewprints of Fig. 12 fall into the second class. Viewprints of the same class are quite similar. This division of viewprints matches the division discussed earlier in connection with the experimental results shown in Figs. 2 and 3. For closely spaced stimuli (separations $\leq 1.5$ min) the dark bars on which the judgment is made are barely visible corresponding to low $d'$ values. As was suggested in our discussion of the middle-spatial-frequency cues, the bisection judgment may be based on comparing the visibility of one or both of the dark bars associated with the two gaps.

**Viewprints for Five-Line Stimulus**

Figure 14 shows viewprints for several five-line stimuli for the C3 and C5 mechanisms. The separation between the test line (before displacement) and the reference lines was always 1.3 min, and the test-line offset was 0.02 min. The separation between the inner reference line and the outer flanking line in the top four pairs of panels was 1.0, 1.2, 1.5, and 2.0 min. The bottom pair of panels is for the case when the flanking lines are not present. The experimental results shown in Fig. 6 indicate that the 0.02-min test offset is slightly below threshold for the three-line case, in agreement with the absence of a differential response in the bottom panel for the C3 mechanism. The introduction of flanking lines causes the differential response to increase, just as was found in our experiments (Figs. 6 and 8 show psychophysical thresholds of about 0.018 min for optimally positioned flanks). For the C3 mechanism the maximum facilitation occurs for flank spacings of 1.2 min. For the C5 mechanism the maximum facilitation occurs for flank spacings of 1.5 min. The tuning of the facilitation effect is quite sensitive to the precise bandwidth and receptive-field shape of the mechanisms. The facilitation data provide a sensitive test of any model attempting to account for hyperacuity. For the experiments described here, both C3 and C5 bandwidths are sensitive to the flanks; however, the broader-bandwidth mechanisms (C3) provide a slightly better match to the experimental data than do the narrower-bandwidth mechanisms.

**DISCUSSION**

The present paper has shown that spatial hyperacuity thresholds may be exquisitely low—even breaking the 1-sec barrier. A framework has been developed to show that these remarkable position-discrimination thresholds do not require mechanisms that are especially sensitive or unusual. Rather, standard mechanisms with tuning characteristics similar to the simple cells described by Hubel and Wiesel are capable of providing cues for detection and discrimination of the direction of offset. The elements of the present model are not new. An early version of the viewprint display (with the Pythagorean sum) had been used to illustrate the effect of spatial probability summation (spatial pooling) for amplitude-modulated and frequency-modulated gratings. The differential response (the $d'$ of a local mechanism’s response to the test pattern minus the mechanism’s response to the reference pattern) had been used to account for the increased visibility of a third harmonic that was due to the presence of a fundamental. A similar model has been used by Wilson and Bergen, Wilson and Gelb, and Wilson and Regan. A new feature of the present model is that the $d'$ difference is taken after the Pythagorean summation.

In the following sections several of the assumptions inherent in the viewprint approach are examined.

**Mechanism Shape**

The viewprint calculations are based on the assumption that the CSF represents the upper envelope of a continuum of mechanisms with different receptive-field shapes and sizes. The basis functions that we chose were the Cauchy functions. These functions have many analytic advantages, which were described earlier. Other filter shapes might give similar results, but the Pythagorean summation would be more difficult since a simple analytic expression for both even- and odd-symmetric receptive fields with the same spatial-frequency response is available only for Cauchy functions.

**Sampling in Size**

The sampling interval in spatial frequency was 0.125-octave steps for the viewprints shown. In our analysis the sampling-interval step size could not be much larger than about 0.3 octave or some of the critical features (null points or asymmetries) needed for the offset discriminations would have been missed. This is an important difference between...
our viewpoint approach and the limited multiple-filter models of Wilson and Bergen\textsuperscript{53} and Wilson and Gelb.\textsuperscript{55} If mechanism sizes were coarsely spaced in the observer's visual system, then we might expect to find peaks and troughs in bisection thresholds as a function of separation similar to the scalloped hue-discrimination function. The only structure in our data occurred at a separation of about 2 min, which we explain as the transition from the detection regime to the discrimination regime. The question of structure in other size-width experiments is currently a matter of controversy.\textsuperscript{50} The finding of reliable structure in size judgments for large interline separations would provide evidence for a limited set of mechanism sizes rather than the continuum assumed in the viewprints.

**Sampling in Space**

The spatial-sampling interval used in constructing the $d'$ contour plots was arbitrarily small. The differential response was sampled at 0.2-min intervals. However, it would be unreasonable to expect the differenting operation needed to calculate the differential response to be so accurate. Therefore in calculating the differential response we have blurred the filter output over a 1-min span. This was accomplished by spatial filtering the mechanism responses using a Gaussian blur function whose standard deviation was 0.5 min. This Gaussian blur that occurs after the contrast response function is equivalent to a limited spatial averaging similar to probability summation. Figures 12 and 13 indicate that, as the separation between the stimulus lines increases, the differential response to a threshold test offset also increases. This would indicate that an ideal observer should do better than 1 part in 60 for the wider separations. However, all receptive fields need not be sampled at the same rate. A more economical scheme would be to sample larger receptive fields more sparsely.\textsuperscript{40,51} If the coarser mechanisms were averaged over a span larger than 1 min, then the peak responses would be diluted in agreement with experiment.

An important difference between our approach and that of Barlow\textsuperscript{3} and Crick et al.\textsuperscript{4} may be that these authors believe that the limiting feature for hyperacuity is the spatial-sampling interval. We, on the other hand, believe that both the sensitivity of the mechanisms, as specified by the CSF, and the sampling govern the limits to hyperacuity. In foveal vision, for example, even if the spatial-sampling interval were 1 sec, smaller offsets could not be detected since the asymmetric stimulation of the mechanisms would be below threshold. On the other hand, it is clear that sampling more sparsely than 1 min would degrade hyperacuity since some localized cues might be missed.

**Normalization**

The problem of defining the contrast of the stimulus was discussed above. Contrast must be calculated because the mechanism sensitivities are normalized by the CSF. Contrast is the appropriate measure for specifying the stimulus since it does not change when the display is viewed through neutral-density filters, in agreement with the data in Fig. 4. Contrast is defined as some measure of the stimulus luminance divided by some measure of the background luminance. The main problem for any localized stimuli is in defining the background luminance. If there had been many lines, forming a grating, then the background luminance would have unambiguously been the mean luminance, which leads to a 200% contrast of the fundamental. As was discussed earlier, this was how we normalized the viewprints. However, since our three-line stimuli were typically less than 3 min in total width, it is likely that nonlinearities at the edges play an important role in setting the level of local adaptation. Our finding that the predicted bisection acuities are in good agreement with the measured acuities implies that the local adaptation level is set by the immediate vicinity of the test line. In order better to measure the characteristics of local adaptation, it would be useful to measure the increment threshold of the luminance of the middle line as a function of the separation of the reference lines (a form of the Westheimer sensitization effect\textsuperscript{15}) and also to measure the CSF in a narrow rectangular patch.

**Bandwidth**

The Cauchy functions used for the viewprints ($n = 3$ and $n = 5$) had broad bandwidths of 1.5 and 2.0 octaves. Narrower bandwidths were also explored but were found to be too sensitive to the presence of flanking lines and not sufficiently sensitive for detecting the offset of a thin line (see Section 2 of Appendix A for the reason medium-bandwidth mechanisms do poorly on thin lines). Also, as the bandwidth becomes narrower the inhibitory flank becomes deeper and the activity that is due to the central line gets suppressed. Since the binocular task requires extremely localized spatial judgments, it is not surprising that broad-bandwidth mechanisms are used. What is perhaps surprising is that the sensitivity of the broad-bandwidth mechanisms is very close to the limit set by the CSF. It is widely assumed\textsuperscript{15,52} that grating detection (CSF) is mediated by mechanisms with medium bandwidths of about 1.25 octaves. However, the presence of mechanisms with differing bandwidths but with similar sensitivities has been implied from experiments on the detection of localized stimuli. Campbell et al.\textsuperscript{5,6} for example, have shown that in order to account for the detection of thin dark lines almost the entire area under the CSF is required. [See Section 2 of Appendix A for this calculation using the CSF given by Eq. (4).] This linear summation implies the presence of broad-bandwidth mechanisms that cover a large fraction of the full CSF, since probability summation over medium-bandwidth mechanisms amounts to incomplete summation of the area under the CSF. Broad-bandwidth mechanisms have also been implicated in suprathreshold phase-discrimination tasks.\textsuperscript{27}

A graphic example of the conflict between broad bandwidths and medium bandwidths is found in Wilson's and Regan's recent attempt to explain spatial-frequency discrimination using the same mechanisms that are responsible for contrast detection.\textsuperscript{40} The problem with any such attempt is that the frequency-discrimination data exhibit broad bandwidths (a large threshold elevation at frequencies more than three times the adapting frequency),\textsuperscript{34} whereas contrast detection exhibits medium bandwidths (minimal threshold elevation at frequencies twice the adapting frequency).\textsuperscript{44} Their "blind" prediction\textsuperscript{49} did not adequately predict the broad tuning of the frequency-discrimination threshold elevation.

This identical point, that mechanisms involved in frequency judgments have broader tuning than the mechanisms used for
contrast judgments, was also reached a decade ago based on the broad extent of the spatial-frequency shift following adaption. The theme of the present analysis is that broad-bandwidth mechanisms may be responsible for size-frequency judgments and that these mechanisms have a sensitivity that is close to the sensitivity of the medium-bandwidth mechanisms responsible for contrast detection.

Paired Mechanisms
The Pythagorean sum in Eq. (9) required an assumption of matched symmetric and antisymmetric mechanisms at each sample point. But matched pairs were used only for mathematical convenience. Whereas other schemes for spatial vision require matched even- and odd-symmetric mechanisms in order to encode phase information, our scheme used the matched pairs in order to eliminate absolute phase information from affecting the differential response values. Any other method for degrading absolute-position information, such as eye movements and pooling over neighboring mechanisms, would work as well. One of the most important implications of our analysis is that, even though position information has been degraded by the Pythagorean sum, the signal-to-noise ratio is still sufficient to account for the bisection thresholds.

One-Bit Phase
The shading in Figs. 12–14 represents the regions where the off-center mechanisms are stimulated, and the unshaded regions correspond to the on-center mechanisms being stimulated. There are both psychophysical and anatomical-physiological reasons to believe that the observer is highly sensitive to this one bit of phase information. Tolhurst and Dealy have compared the psychophysical threshold for detecting thin lines and edges with the threshold for discriminating the polarity of the lines (light versus dark) and edges (right versus left). They found that the polarity could be discriminated at contrasts just above the detection threshold. The polarity of the detection mechanisms is labeled should not be surprising in light of the recent anatomical and physiological evidence for clear segregation of on-center and off-center cells into different layers of the retina and the lateral geniculate nucleus. It is reasonable to expect that this segregation will also be present at the cortex. The presence of one-bit phase may not be useful for high-quality hyperacuity judgments, but it would be ideal for discriminating the polarity of lines and edges. One-bit phase would also be useful for detecting the displacement of a sinusoidal grating. The sensitivity of mechanisms for detecting grating displacements is discussed in Section 4 of Appendix A.

Response Strategies
In our analysis the differential response was the basis for the discrimination judgment. The differential response is the response of a mechanism to the test pattern (with a displaced test line) minus the response to the reference pattern (with an undisplaced test line). This scheme requires the observer to superimpose the two responses accurately in order to perform the subtraction. We do not believe that the visual system actually makes use of the differential response in this way. Rather, we believe that an accurately labeled feature (such as the frequency of the null point or the strength of opponent activity invoked by Regan and Beverley) is extracted from the viewprint and compared with the memory for that feature in the reference viewprint or compared with the mirror-image feature simultaneously present in the viewprint of the bisection stimulus. The importance of the differential response is that it places an upper limit on the strength (signal-to-noise ratio) of the particular feature. Neither the spatial-frequency cues nor spatial-position cues are usable if the differential response is below threshold.

It is quite likely that the human visual system is exquisitely good at remembering the label of a feature. Our bisection experiments did not require memory since the mirror-image reference feature was present simultaneously. However, in two-line width judgments that did require a memory comparison, the Weber fractions of 1/40 to 1/50 were almost as good as the 1 part in 60 of the present bisection thresholds.

SUMMARY
The viewprint approach shows that standard mechanisms are capable of providing cues to very small spatial offsets. The viewprints show that mechanisms with sensitivity set by the CSF, and spaced no closer than the separation of foveal cones, are quite capable of signaling the presence of a threshold-level position change even under conditions in which the human observer can discern direction of offsets of less than 1 sec. In the experimental section of this paper we stressed the presence of different cues to bisection hyperacuity. One of the great advantages of the viewprint approach is that it presents a large number of cues for offset discrimination and shows which cues are likely to be used in making the judgments. Thus the viewprint approach appears to work both for closely spaced stimuli (for which the presumptive cue is based on brightness differences) and for more widely spaced stimuli (i.e., the traditional hyperacuity regime). Appendix A provides insight into why this approach succeeds. Section 1 of the appendix suggests why the optimal hyperacuity limit is about 0.02 min. Sections 3 and 5 of the appendix suggest that size judgments should be accurate to about 1 part in 30 in the Weber regime.

The essence of the viewprint approach is that optimal relative-position information is obtained from mechanisms tuned to size. It might well be that the position of a mechanism within a hypercolumn is not labeled so accurately as its size. This notion of size coding is central to the spatial-frequency approach to vision. The viewprint scheme, which uses size coding on a local scale and a Pythagorean sum to eliminate absolute phase, is especially useful in the presence of eye movements that would otherwise degrade the image.

The viewprint computation, like the human brain, is a multistage operation. Undoubtedly, some of the proposed stages will require additional refinement. Greater precision in comparing experiment with theory must await measurements of the contrast-response function using stimuli that resemble those used in the bisection task and measurement of bisection thresholds using lines of different contrasts.

Our hypothesis is that with practice (and feedback) the observer learns to read the cues by attending to a small subset of optimal mechanisms (in much the same way as the reader sees the cues in the viewprints presented in Figs. 12–14). This approach provides a viable quantitative account for bisection hyperacuity. How well it will account for more-complex
two-dimensional stimuli, such as vernier acuity with pairs of chevrons, remains to be tested.

APPENDIX A: SIMPLIFIED SPATIAL-FREQUENCY ANALYSIS

The goal of this appendix is to provide some insight into why the contrast-response function worked so well in predicting the bisection thresholds. We show here that the CSF sets the bisection thresholds for small separations and the contrast-discrimination function sets the bisection thresholds in the Weber regime. Instead of using viewprints, this section will use a simplified spatial-frequency analysis.

1. Detection of a Subharmonic

One way to simplify a spatial-frequency analysis is to use a stimulus with only a few Fourier components. Typical hyperacuity stimuli, however, have their critical features localized in a very small spatial region and therefore have their spatial frequencies spread broadly over a wide continuum. The continuum can be replaced by a discrete spectrum if repeating hyperacuity stimuli are used. Repeating a pattern has a minor effect on the local spatial features but greatly simplifies the Fourier domain by concentrating the spectrum at a few points. Thus if a three-line bisection stimulus with a separation of 4/3 min is repeated every 4 min, a grating with a 4-min repetition cycle is obtained. A 4-min repetition cycle implies that the relevant Fourier components of the stimulus are at 15, 30, and 45 c/deg. The fourth and higher harmonics are at and above 60 c/deg and therefore will not contribute to the visibility of the pattern.

When the test line is not displaced, the contrasts of the first three harmonics are 0, 0, and 200%, respectively, since the 45-c/deg grating is composed of thin equispaced white lines on a dark background. When the repeating test lines are displaced by 0.03 min, the contrasts are 2, 4, and 200% for the first, second, and third harmonics, respectively, as obtained by Fourier analysis of the stimulus. At 15 c/deg the contrast threshold is about 2%.29 so a displacement of 0.02 min should just be visible. The viewprints in Fig. 14 for the stimulus with lines at −2.5, −1.3, 0.02, 1.3, and 2.5 show a similar behavior. There is a group of mechanisms tuned to spatial frequencies below the pattern’s frequency that are not stimulated after the displacement. These mechanisms are ideally positioned to detect the subharmonics that are generated when the test line is displaced away from the bisection point.

Discriminating the direction of the offset is another matter. A mechanism centered at 20 c/deg would be sensitive to both the 15- and the 30-c/deg components and could signal the relative phase (right versus left sawtooth) of the first and second harmonics and thereby discriminate the direction of the 0.02-min offset. However, before one calculates whether the 20-c/deg channel is sufficiently sensitive to account for the direction discrimination, it should be noted that it is quite unlikely that the offset direction could actually be discriminable for the repetitive pattern. In the three-line bisection stimulus the observer had to decide which of the two dark bars (between the three bright lines) was wider. For the repetitive pattern introduced here, the observer sees a multitude of dark bars of three different sizes and must discriminate their sequence during a brief exposure—a much more difficult task than the three-line experiment.

2. Detection of a Thin Line

In the discussion of experimental results, it was pointed out that for our optimal case (Fig. 6) the bisection cue was based on detecting and localizing a thin dark line on a narrow (5-min) background. The bisection threshold of 0.85 sec is close to the threshold of 0.5 sec for detecting a single dark line on a uniform field.14 We now show that this high sensitivity to thin lines required broad-bandwidth mechanisms. Suppose that the CSF is composed of a single broad-bandwidth mechanism whose tuning is given by Eq. (4):

$$\text{CSF}(f) = A(p^2 + 1)(\exp(-\rho f))$$

where $A = 600$ and $\rho = 1.25$ min/deg = 0.13 deg/cycle. The integral of the CSF specifies the sensitivity to a thin line.52 The integral (a gamma function) equals $A (0.5!) = 4000$ deg$^{-1}$. This sensitivity predicts that a black line whose width is 3600/4000 = 0.9 sec should be at threshold. The higher-sensitivity found by Hecht and Mintz14 must be attributable to a higher CSF or a slightly looser criterion. It is impossible to account for the visibility of a thin line in terms of multiple medium-bandwidth mechanisms. Probability summation cannot be stronger than the total linear summation that is done by the broad-bandwidth filter. Probability summation can operate only on independent nonoverlapped mechanisms, since overlapped mechanisms would have correlated noise. The high visibility of thin lines requires nearly the full strength of the CSF,52 thereby supporting the existence of broad-bandwidth mechanisms such as the C3 Cauchy function with sensitivity almost equal to the CSF. The existence of such mechanisms would not be clearly revealed by subthreshold summation studies if the mechanism sensitivity were about 25% lower than the CSF, nor by adaptation and masking studies, which reveal properties of the narrowest mechanisms.

This Fourier-domain analysis showed that the very small 0.02-min thresholds found in our experiment were compatible with the CSF with minimal assumptions. The viewprint analysis considered earlier was based on this connection between detection (CSF) and displacement. The next sections consider lines and gratings with wider spacing, where hyperacuity thresholds become linked to the Weber’s law contrast-increment discrimination.

3. Weber Regime: Amplitude and Frequency Modulation

The purpose of this section is to provide evidence linking the threshold for size discrimination to the threshold for local contrast discrimination. This section will consider the data of Jamar et al.56 and Arend and Lange,57 which compared the detection of amplitude modulation (AM) and frequency modulation (FM) of a suprathreshold carrier sinusoidal grating.

Consider their two stimuli: $\cos(fx) + E/2[\cos(f - m)x \pm \cos(f + m)x]$. The peaks-add case is an AM pattern, which can be written as $\cos(fx)(1 + \epsilon \cos(mx))$. Both studies found that for the AM phase threshold was $\epsilon \approx 0.1$, corresponding to a 20% difference between the high and low contrasts. Arend and Lange examined $f/2\pi = 10$ c/deg and $m/2\pi = 5$ c/deg, whereas Jamar et al. examined many combinations of $f$ and $m$. The frequencies used by Arend and Lange are a first plus a second plus a third harmonic in which every other bar alternates in size or contrast.
The peaks-subtract phase can be rewritten approximately as an FM pattern:
\[
\cos(fx) + \epsilon \sin(fx) \sin(mx) \approx \cos[fx - \epsilon \sin(mx)].
\]

The approximation is based on using the trigonometric identity \(\cos(A - B) = \cos A \cos B + \sin A \sin B\) and realizing that, for small \(\epsilon\), \(\cos[\epsilon \sin(mx)] \approx 1\) and \(\sin[\epsilon \sin(mx)] \approx \epsilon \sin(mx)\). For the case when \(m = f/2\), the stimulus is best thought of as a series of dark bars with alternate spacings of \(2(\pi + \epsilon)/f\) and \(2(\pi - \epsilon)/f\). That is, the distance between black bars changes by \(\epsilon/\pi\) for \(\epsilon = 0.1\) compared with the unmodulated case. This stimulus is in fact a repeating version of our three-line bisection stimulus (with reversed polarity). In both studies\(^{56,57}\) it was found that the FM threshold for \(m = f/2\) is only slightly greater than the AM threshold. Thus the mechanisms responsible for detection of the FM (a hyperacuity judgment) are not mechanisms with unusually sensitive properties. They need in fact be only slightly less sensitive than the mechanisms used for detecting contrast changes. There is no need to postulate special mechanisms.

This section shows the relationship between local changes in contrast and local changes in size. A similar relationship can be shown between uniform changes in contrast and uniform shifts in phase. This is dealt with next.

4. Weber Regime: Contrast and Phase Increments

The simplest connection between contrast and phase occurs for a single-component pattern. Consider the one-component pattern in which a grating of contrast \(c\) is given an increment:
\[
c[\cos(fx) + 0.1 \cos(fx + \theta)].
\]

For \(\theta = 0\), the contrast is increased to 1.1 cycles. For \(\theta = \pi/2\), the second term produces a phase shift of 0.1/2\(\pi\), or about 1/60 cycle (a 6°-deg phase shift). We postulate that for the transient detection strategy (low spatial frequency, rapid temporal onset) the visibility of the second term is independent of the phase \(\theta\). These values are in good agreement with the findings of Nakayama and Silverman.\(^58\) They find that in the Weber regime the displacement threshold is about 1/60 cycle, which agrees with the 10% contrast jnd for a full grating (compared with the 20% value for local contrast).\(^{36,37}\) A similar limiting value for the phase threshold was found in the present bisection experiments and in the phase-discrimination experiments of Badcock,\(^37\) in which a harmonic spatial frequency acts as the reference.

In describing their results, Nakayama and Silverman\(^58\) introduce the concept of "effective contrast." Effective contrast can be defined as the contrast that, when multiplied by the Weber fraction, equals the threshold contrast. Thus effective contrast equals the threshold times the inverse Weber fraction. The effective contrast for our contrast-response function is given by \(S/\Delta S = 1/(1 + W + 1/\Delta S)^{1/2} - 1\) and is shown in Fig. 10D as the dotted line. The stimulation \(S\) equals the contrast [see the discussion following Eq. (8)]. The threshold contrast has been normalized to unity. In the effective-contrast approach, the threshold is fixed and the effective contrast saturates at a low level. In the contrast-response-function approach, the threshold increases as the background contrast increases, in accordance with Weber's law. These two approaches are mathematically equivalent.

This neat relationship between contrast discrimination and displacement detection holds only at low spatial frequencies. At higher spatial frequencies, the constant eye movements would be expected to mask the observer's ability to detect unrefereced displacements while not masking the contrast judgment. Indeed, above 3 c/deg the displacement threshold is a constant visual angle (≈10 arc sec) rather than a constant phase angle.\(^59\)

5. Weber Regime: The Steep Slopes of the Mechanism's Tuning Function

A reasonable model for size judgments uses the steep portion of the mechanism's frequency-tuning curves, in analogy to color-discrimination models.\(^{44,45}\) It is useful to calculate the implications of this model for our contrast-response function. In the Weber regime the contrast-response function [Eq. (1)] becomes
\[
d' = 5 \ln S(\sigma f),
\]
where the factor of 5 comes from the 20% contrast jnd and
\[
S(\sigma f) = (\sigma f)^n \exp(-\sigma f).
\]

A small change in \(d'\) can be obtained by combining Eqs. (A1) and (A2):
\[
\Delta d' = 5 \Delta S/S = 5 \Delta f/(n/f - \sigma).
\]

For the mechanism whose peak is at frequency \(p\), \(\sigma\) is given by \(\sigma = n/p\) and Eq. (A3) becomes
\[
\Delta d' = 5n(\Delta f/(p - f)/p.
\]

The quantity \((p - f)/p\) is the fractional distance from the peak of the tuning curve to the point at which the frequency judgment is made. As one goes farther from the peak the tuning curve becomes steeper, which should lead to a more-sensitive frequency discrimination. However, at some point the transition between the saturating and the threshold portions of the contrast-response function is reached. We assume that distances larger than 2 standard deviations away from the peak lead to unreliable responses and are not helpful. Thus an assumption is made that the 2-standard-deviation point of the tuning curve is used for the frequency-discrimination judgment. Since the standard deviation of the Cauchy function is given by \(\sqrt{(n + 1)/n}\), the quantity \((p - f)/p\) equals \(2[\sqrt{(n + 1)}]/n\). Using this quantity in Eq. (A4) and assuming that \(\Delta d' = 1\) for the discrimination jnd gives the following expression for the just-noticeable frequency change:
\[
\Delta f/f = 0.1/\sqrt{(n + 1)}.
\]

Thus a broad-bandwidth mechanism \((n = 3)\) leads to a 5% frequency discrimination, and a medium-bandwidth mechanism \((n = 15)\), corresponding to Wilson's and Gelb's high-spatial-frequency filter,\(^43\) leads to a 2.5% frequency discrimination. As we discussed above, it is unlikely that medium-bandwidth filters are appropriate for the bisection task since they would be excessively disturbed by flanking lines.
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