RELATIONSHIPS BETWEEN KERNELES MEASURED WITH DIFFERENT STIMULI

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INTRODUCTION

The purpose of this paper is to clarify the interrelationships between different functional expansions of a given nonlinear system. Suppose, for example, the stimulus, \( x(t) \) was used to measure the Wiener kernels of a system. If a different stimulus were used (say with a higher mean level or a higher power density), then a different set of kernels would be obtained. This paper presents formulas and examples relating the two sets of kernels. Many of the results to be presented here were published several years ago (Klein and Yasui, 1979; Klein, 1979). However, those papers did not provide sufficient examples. The Klein–Yasui paper may in fact have misled the reader by limiting the scope of the applications. The present article seeks to clarify the previous results.

This introduction will give a flavor and overview of our enterprise by examining how the kernels of a linear system depend upon the mean luminance of the stimulus (speaking of visual stimuli). For a linear system the functional expansion can be written in discrete time as

\[
y(t) = h_0^L + \delta \sum_\tau h_1^L(\tau) \left( x^L(t - \tau) - L \right)
\]

where the superscript, \( L \), is the mean luminance of the stimulus (\( L = \bar{x}^L(t) \)), where the bar indicates time average over the data-record, and \( h_0^L \) is the average response to the stimulus \( x^L(t) \) given by

\[
h_0^L = \bar{y}(t)
\]

The sampling interval, \( \delta \), is associated with a summation of discrete samples just as \( dt \) would be associated with an integration of continuous samples.

Suppose that instead of using the stimulus \( x^L(t) \), the system is tested with a stimulus \( x^{L'}(t) = x^L(t) + (L' - L) \). The first order kernel, \( h_1(\tau) \) would not be changed since the system is linear, so a superscript on \( h_1 \) will be omitted. The zeroth order kernel, however, would be different and given by

\[
h_0^{L'} = h_0^L + (L' - L) \delta \sum_\tau h_1(\tau)
\]

For small changes \( (L' - L) \), this equation indicates that the rate of change of \( h_0 \) with respect to luminance is given by the sum over the entire first order kernel. In Eq.(20) we
show a similar connection between the luminance dependence of $h_1(r)$ and a summation over the second-order kernel.

In this paper we consider not only changes in mean luminance but also arbitrary changes in the stimulus. A stimulus can be characterized by its cumulants (or moments) such as the power density, skewness, kurtosis, etc. The formalism to be developed will show how the system kernels depend upon the stimulus cumulants in addition to depending upon the system nonlinearities.

The formulae to be developed simplify for Gaussian stimuli and Poisson stimuli. The connection between Poisson stimuli and Gaussian stimuli are especially interesting because they allow the Wiener kernels associated with Gaussian stimuli to be related to the Poisson kernels obtained with transient brief flashes.

**EXCLUSIVE DIAGONALS EXPANSIONS**

The output, $y(t)$, of an analytic, time invariant, nonlinear, stable system can be related to the input, $x(t)$, by a functional expansion:

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

where the terms $y_n(t)$ represent orthogonal (Wiener) functionals.

All time intervals will be digitized with a sampling time $\delta$, and a bar over a quantity indicates the time average of that quantity. The first two terms have the linear form shown in Eq.(1). The terms in Eq.(3) for $n > 1$ are the nonlinear contributions. The general term $y_n(t)$ is most usefully written in terms of single counting (SC) summations.

$$y_n(t) = \sum_{k=1}^{SC} \sum_{n_j} H_n(r_1^{n_1} \ldots r_k^{n_k}) \delta^n \prod_{i=1}^{k} X_{n_i}(t - \tau_i)$$

(4)

The notation $r_j^{n_j}$ means that the time delay $\tau_j$ occurs $n_j$ times. The single counting (SC) means that each particular combination of time delays only occurs once in the summation. The single counting constraint can be illustrated by considering the case $n = 3$:

$$y_3(t) = \sum_{r_1 \neq r_2} \delta^3 H_3(r_1, r_2, r_3) X_1(t - r_1) X_1(t - r_2) X_1(t - r_3) + \sum_{r_1 \neq r_2 \neq r_3} \delta^3 H_3(r_1^2, r_2, r_3) X_2(t - r_1) X_1(t - r_2)$$

$$+ \sum_{r_1 \neq r_2 \neq r_3} \delta^3 H_3(r_1^3) X_3(t - r_1)$$

(5)

The function $X_n(t)$ is an $n$’th order polynomial of $x(t)$. The coefficients of the polynomial are constructed by a Gram-Schmidt orthogonalization procedure. The orthogonality conditions are

$$X_n(t) X_m(t - \Delta) = 0 \text{ if } n \neq m \text{ or } \Delta \neq 0$$

In order to ensure orthogonality for $\Delta \neq 0$ the stimulus $z(t)$ is chosen such that the stimulus at one time is independent of the stimulus at any other time and leads to
where \( f(t) \) and \( g(t) \) are functions of \( z(t) \). A signal, \( z(t) \), which satisfies Eq. (7) can be called a “white noise” stimulus.

The orthogonality (6) of the functions \( X_n(t) \) can be ensured for \( \Delta = 0 \) by the condition (Barrett, 1963; Klein and Yasui, 1979):

\[
X_n(t) = \frac{\det[M_n(x)]}{\det[z^{n-1}M_{n-1}(x)]}
\]

where

\[
M_n(x) = \begin{bmatrix}
1 & \bar{z} & \bar{z}^2 & \cdots & \bar{z}^n \\
\bar{z} & \bar{z}^2 & \bar{z}^3 & \cdots & \bar{z}^{n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{z}^{n-1} & \bar{z}^n & \bar{z}^{n+1} & \cdots & \bar{z}^{2n-1} \\
1 & z(t) & z^2(t) & \cdots & z^n(t)
\end{bmatrix}
\]

The functions \( X_n(t) \) form an orthogonal basis.

The first two basis functions are

\[
X_1(t) = z(t) - \bar{z}
X_2(t) = z^2(t) - z(t)(\bar{z}^3 - \bar{z}^2\bar{x}) / (\bar{z}^2 - \bar{z}^2) - (\bar{z}^3 - \bar{z}^2)(\bar{z}^2 - \bar{z}^2) / (\bar{z}^2 - \bar{z}^2)
\]

To prove \( X_n(z)X_m(z) = 0 \) for \( m < n \) we must simply show \( z^mX_n(z) = 0 \) for \( m < n \). This last equation is true since the bottom row of \( \bar{z}^mM_n \) is equal to the \((m+1)^{th}\) row which causes the determinant to vanish.

It is convenient to define a set of analysis functions \( V_n(t) \) which are proportional to \( X_n(t) \), \( V_n(t) = kX_n(t) \), but are normalized so that

\[
\overline{V_m(t)V_n(t)} = 0 \quad \text{if} \quad m \neq n
\]
\[
= 1 \quad \text{if} \quad m = n
\]

The functions, \( V_n(t) \) are called analysis functions because they will be used in the cross-correlation analysis to calculate the kernels. The expansion functions \( X_n(t) \) are used to obtain a model response. The analysis functions are given by

\[
V_n(t) = \frac{\det[M_n(x)]}{\det[z^nM_n(x)]}
\]

For example:

\[
V_1(t) = \frac{(z(t) - \bar{z})}{(\bar{z}^2 - \bar{z}^2)}
\]

In term of the analysis functions, \( V_n(t) \), the Wiener expansion (3) and (4) can be easily inverted:

\[
\delta^nH_n(\tau_1, \cdots, \tau_k) = y(t) \prod_{i=1}^k V_n(t - \tau_i)
\]
The kernels defined by (14) and (4) have a normalization which is slightly different from the usual Wiener kernels $H_n$ defined by Lee and Schetzen (1965):

$$n! h_n(\tau_1^{n_1} \cdots \tau_k^{n_k}) = \prod_i (n_i)! H_n(\tau_1^{n_1} \cdots \tau_k^{n_k})$$

(15)

**CHANGE OF BASIS**

Suppose a set of kernels, $H_n^x$, have been obtained by using a stimulus $x(t)$ (indicated by the superscript). Or a set of kernels, $H_n^w$, could have been obtained using a stimulus $w(t)$. In this section the relationship between $H_n^x$ and $H_n^w$ is examined.

As long as the stimuli $x(t)$ and $w(t)$ are sufficiently similar (to be discussed in the next section) it is possible to express the kernels $H_n^w$ in terms of the kernel $H_n^x$. We should first introduce a notation which allows one to express the response to a stimulus $w(t)$ which is different from the stimulus $x(t)$, originally used in measuring the kernels. Eq. (4) can be rewritten,

$$y_n(t) = \sum_{k,n} \sum_{r_j} H_n^w(\tau_1^{n_1} \cdots \tau_k^{n_k}) \delta^n \prod_{i=1}^k X_n^w(w(t - \tau_i))$$

(16)

where

$$X_n^w(w(t)) = \frac{\det [M_n^x(w)]}{\det [x^{n-1}M_{n-1}(x)]}$$

(17)

and $M_n^x(w)$ is the same as $M_n(x)$ defined in (9) except that $x(t)$ is replaced by $w(t)$ in the bottom row of the matrix. The analysis functions $V_n(t)$ are defined just as in (12) except $x$ is replaced by $w$ everywhere, not just in the bottom row. For example:

$$X_1^w(w(t)) = w(t) - \bar{w}$$

$$V_1^w(w(t)) = (w(t) - \bar{w})/(\bar{w}^2 - \bar{w}^2)$$

The relationship of $H^w$ to $H^x$ is directly obtained by combining (14), (16) and (17):

$$H_n^w(\tau_1^{n_1} \cdots \tau_k^{n_k}) = \sum_{p,r,m} \prod_{i=1}^p C^{x,w}_{m,i} \sum_{r_j} \delta^{p-k} H_n^x(\tau_1^{m_1} \cdots \tau_p^{m_p})$$

(18)

where

$$C^{x,w}_{m,i} = \frac{X_n^x(w) V_n^w(w) \delta^{m-n}}{X_n^w(w)}$$ for $n \neq 0$$

$$C^{x,w}_{m,0} = \frac{X_n^x(w)}{X_n^w(w)} \delta^{m-1}$$

In the language of Klein and Yasui (1979) the summations over $r_j$ are exclusive summations since the diagonal elements are not included. For the special case in which both $x(t)$ and $w(t)$ are symmetric ($\bar{x}^n = \bar{w}^n = 0$ if $n$ is odd) the expansion simplifies:
\begin{align}
H_1^w(\tau_1) &= H_1^z(\tau_1) + \left( \frac{w^4}{w^2} - \frac{z^4}{x^2} \right) \delta^2 H_5^z(\tau_1^3) \\
&\quad + (\overline{w^2} - \overline{z^2}) \delta \sum_{\tau_2 \neq \tau_1} \delta H_4^z(\tau_1, \tau_2^3) + O(H_6) \tag{19a}
\end{align}

\begin{align}
H_2^w(\tau_1, \tau_2) &= H_2^z(\tau_1, \tau_2) + \left( \frac{w^4}{w^2} - \frac{z^4}{x^2} \right) \delta^2 \left[ H_4^z(\tau_1, \tau_2^3) + H_4^z(\tau_2, \tau_2^3) \right] \\
&\quad + (\overline{w^2} - \overline{z^2}) \delta \sum_{\tau_2 \neq \tau_1} \delta H_4^z(\tau_1, \tau_2, \tau_2^3) + O(H_6) \tag{19b}
\end{align}

\begin{align}
H_2^w(\tau_1^2) &= H_2^z(\tau_1^2) + \left( \frac{w^4 - w^2 \overline{w^2}}{w^2 - \overline{w^2} - \frac{x^2}{x^2} - \frac{x^2}{x^2}} \right) \delta^2 H_4^z(\tau_1^3) \\
&\quad + (\overline{w^2} - \overline{z^2}) \delta \sum_{\tau_2 \neq \tau_1} \delta H_2^z(\tau_1^2, \tau_2^3) + O(H_6) \tag{19c}
\end{align}

Note that the terms with summations have a coefficient \((\overline{w^2} - \overline{z^2})\delta\). This coefficient is the difference in power density of the two stimuli since \(P^z = \overline{z^2}\delta\) and \(P^w = \overline{w^2}\delta\). The dependence of kernels on power density will be explored further.

**INCLUSIVE DIAGONAL EXPANSIONS**

A general expansion of one set of kernels in terms of a different set (measured using test stimuli with different statistics) was given by Eq. (18). The expansion coefficients \(C_{nm}\) were relatively straightforward to calculate because the diagonal elements were kept separate from the off-diagonal element. However, in the standard approaches to functional expansions the diagonal terms are included in the integrations over off-diagonal elements. The difference between the diagonal coefficients and the extrapolation of off-diagonal coefficients to the diagonal must be added as additional terms. It was shown in Klein and Yasui (1979) that the low order additional terms are closely related to the cumulants of the test stimuli.

In order to make contact with the usual kernel expression Eq. (19a) can be rewritten in terms of \(h_n()\):

\begin{equation}
H_1^w(\tau_1) = H_1^z(\tau_1) + \left( \frac{w^4}{w^2} - \frac{z^4}{x^2} \right) \delta^2 H_5^z(\tau_1^3) + (\overline{w^2} - \overline{z^2}) \delta \sum_{\tau_2 \neq \tau_1} \delta H_4^z(\tau_1^2, \tau_2^3) \tag{20}
\end{equation}

The factor 3 arises because in our convention the combination \(\tau_1 \tau_2^3\) occurs once whereas in the usual summation convention, that diagonal occurs three times: \((\tau_1 \tau_1 \tau_2), (\tau_1 \tau_2 \tau_1)\) and \((\tau_2 \tau_1 \tau_1)\).

The diagonal element can be included in the summation of Eq. (20) by adding the term \(3(\overline{w^2} - \overline{z^2}) \delta^2 h_3(\tau_1^3)\).

\begin{equation}
H_1^w(\tau_1) = H_1^z + \left[ Q_{13}(w) - Q_{13}(z) \right] H_5^z(\tau_1^3) + \left[ Q_{02}(w) - Q_{02}(z) \right] 3 \sum_{\tau_2} \delta H_4^z(\tau_1 \tau_2^3) \tag{20'}
\end{equation}

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where \( Q_{13}(z) = \left( z^4 - 3z^2 \right) \delta^2 / z^3 \) and \( Q_{02}(z) = z^2 \delta \)

The summation label \( IN \) stands for inclusive summation.
We conjecture that the general expansion, with no restriction on \( z(t) \) or \( w(t) \) is:

\[
H_m^n(\tau_1^{n_1} \cdots \tau_k^{n_k}) = \sum_{r} \frac{1}{r!} \left( \sum_m \prod_i (Q_{m_i}(w) - Q_{n_i}(w)) \right) \sum_{i > k} H_m^m(\tau_1^{m_1} \cdots \tau_k^{m_k}) \delta^r \quad (21)
\]

The kernels \( H \) are used rather than \( h \) to avoid unnecessary factorials. The summation includes diagonal elements. The low order coefficients are related to cumulants as discussed by Klein and Yasui (1979). The conjecture is based on examining Eq. (21) for special cases as discussed below.

In the reminder of this paper we focus on Gaussian stimuli and Poisson stimuli. These are the only two classes of stimuli for which the inclusive summations are much simpler than the exclusive summations given by Eq. (18). For all other stimuli, the inclusive summations become much more complicated than Eq. (18).

**COMPARISON OF GAUSSIAN KERNELS FOR DIFFERENT POWER DENSITY**

The first example will be to show how kernels measured with Gaussian stimuli of power density \( P \), are related to kernels measured using Gaussian stimuli of power density \( P_0 \). Kernels measured using Gaussian stimuli will be called Gaussian kernels. Kernels measured using stimuli (Gaussian or non-Gaussian) with negligible power density will be called Volterra kernels.

For Gaussian stimuli all the \( Q_{mn}(z) \) vanish except for \( Q_{02}(z) = P \). Eq. (21) can then be written as (Klein, 1979):

\[
H_m^n(\tau_1^{n_1} \cdots \tau_k^{n_k}) = \sum_r \frac{(P - P_0)^r}{r!} \left( \sum_{i > k} H_m^m(\tau_1^{m_1} \cdots \tau_k^{m_k}) \delta^r \right) \quad (22)
\]

If \( P_0 \neq 0 \), the kernels \( H_m^m(\tau \ldots) \) become Volterra kernels, \( G_m(\tau \ldots) \) and Eq. (22) becomes the expansion of Gaussian kernels in terms of Volterra kernels. The first few terms in the expansion are:

\[
H_1^{GP}(\tau_1) = G_1(\tau_1) + P \sum_{\tau_2} \delta G_3(\tau_1, \tau_2^2) + \frac{p^2}{2!} \sum_{\tau_2, \tau_3} \delta^2 G_5(\tau_1, \tau_2^2, \tau_3^2) + O(G_7) \quad (23)
\]

If, on the other hand, \( P = 0 \) in Eq. (22), an expansion of Volterra kernels in terms of Gaussian kernels is obtained:

\[
G_1(\tau_1) = H_1^{GP}(\tau_1) - P_0 \sum_{\tau_2} \delta H_3^{GP}(\tau_1, \tau_2^2) + \frac{p^2}{2!} \sum_{\tau_2, \tau_3} \delta^2 H_5^{GP}(\tau_1, \tau_2^2, \tau_3^2) + O(H_7) \quad (24)
\]

Equations (23) and (24) have been derived independent of the conjectured general relation in Eq. (21).
DEPENDENCE OF GAUSSIAN KERNELS UPON MEAN LUMINANCE AND POWER DENSITY

Equation (21) also applies to the case in which the stimuli are not zero mean. The stimulus mean, $\bar{x}$, will be called $L$, the mean luminance for the case in which $x(t)$ is a visual stimulus. The only non-vanishing cumulants are $Q_0(x) = L$ and $Q_2(x) = P$. All other cumulants vanish because the stimuli are Gaussian. Eq. (21) becomes:

$$H_n^{GPL}(\tau_1^{n_1} \cdots \tau_k^{n_k}) = \sum_{r \geq 1} \frac{(P - P_0)^r}{r!} \frac{(L - L_0)^s}{s!} \sum_{\delta > 0} \delta^{r+s} H_m^{GPLL_0}(\tau_1^{n_1} \cdots \tau_k^{n_k} \tau_{k+1}^2 \cdots \tau_{k+r}^2 \tau_{k+r+1}^r \cdots \tau_{k+r+s}^r) (25)$$

The leading contributions to the dependence of the first order kernel upon luminance are:

$$H_1^{GPL}(\tau_1) = H_1^{GPLL_0}(\tau_1) + (L - L_0) \sum_{\tau_1} \delta H_2^{GPLL_0}(\tau_1 \tau_1) + \frac{(L - L_0)^2}{2!} \sum_{\tau_1, \tau_2} \delta^2 H_3^{GPLL_0}(\tau_1 \tau_2 \tau_2) (26)$$

or

$$\frac{dH_1^{GPL}(\tau_1)}{dL} = \sum_{\tau_1} \delta H_2^{GPLL_0}(\tau_1 \tau_1)$$

COMPARISON OF NON-GAUSSIAN KERNELS TO GAUSSIAN KERNELS OF THE SAME POWER DENSITY

We now examine the relationship of kernels gotten with a zero mean non-Gaussian stimulus, to the zero mean Gaussian kernels of the same power density. The derivation of this Wiener expansion is given by Klein and Yasui (1979).

$$H_n(\tau_1^{n_1} \cdots \tau_k^{n_k}) = \sum_{r, m_1} \frac{1}{r!} \sum_{\tau_1} \delta^r H_m^G(\tau_1^{m_1} \cdots \tau_k^{m_r}) \prod_{i=1}^n Q_{nm_i} (27)$$

where the summation over $m_i$ is constrained by $m_i \geq n_i$ for $i \leq k$ and $m_i > 3$ for $i > k$. The factors $Q_{nm_i}$ involve sums of moments of $x$. The first few $Q_{nm}$ are given by:

$$Q_{1n} = 1 \quad Q_{01} = \bar{x} = 0$$
$$Q_{02} = \bar{x}^{\bar{x}} \delta \quad Q_{03} = \bar{x}^{\bar{x}} \delta^2 \quad Q_{04} = (\bar{x}^{\bar{x}} - 3\bar{x}) \delta^2 \quad Q_{05} = (\bar{x}^{\bar{x}} - 10\bar{x}^{\bar{x}}) \delta^4$$
$$Q_{1n} = Q_{0n+1} / (\bar{x}^{\bar{x}})$$
$$Q_{23} = (\bar{x}^{\bar{x}} - \bar{x}^{\bar{x}}) \delta / (\bar{x}^{\bar{x}} - \bar{x}^{\bar{x}})$$

The low order terms of the Wiener expansion are:
\[ H_1(\tau_1) = H_1^G(\tau_1) + \frac{x_1^3 \delta}{x_1^2} H_2^G(\tau_1^2) + \left( \frac{x_1^3}{x_1^2} - \frac{x_1^3}{x_1^2} \right) \delta^2 H_3^G(\tau_1^3) + \sum_{2} \delta H_4^G(\tau_1^2) x_1^3 \delta^2 \ldots \]

\[ H_2(\tau_1 \tau_2) = H_2^G(\tau_1 \tau_2) + \frac{x_1^3 \delta}{x_1^2} \left[ H_2^G(\tau_1^2 \tau_2) + H_3^G(\tau_1 \tau_2^2) \right] + O(H_4) \]  

(29)  

\[ H_2(\tau_1^2) = H_2^G(\tau_1^2) + \left( \frac{x_1^3}{x_1^2} - \frac{x_1^3}{x_1^2} \right) \delta^2 H_3^G(\tau_1^3) + O(H_4) \]

We shall focus on three stimuli: Gaussian, binary, and Poisson. Expectation values of the first few moments of these stimuli are given in Table 1:

**TABLE 1**

<table>
<thead>
<tr>
<th>Gaussian</th>
<th>Binary (zero mean)</th>
<th>Poisson (zero mean)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allowed values</td>
<td>all x</td>
<td>x=L/p</td>
</tr>
<tr>
<td>Distribution</td>
<td>L</td>
<td>L</td>
</tr>
<tr>
<td>(\overline{x}^2)</td>
<td>(\sigma^2)</td>
<td>(L^3/p)</td>
</tr>
<tr>
<td>(\overline{x}^3)</td>
<td>(3\sigma^2)</td>
<td>(L^4/p^3)</td>
</tr>
<tr>
<td>(\overline{x^n})</td>
<td>0 if n odd (n!\sigma^n)</td>
<td>(L^n/p^{n-1})</td>
</tr>
<tr>
<td>Power=</td>
<td>(\sigma^2 \delta)</td>
<td>(L^2(\Delta-\delta))</td>
</tr>
</tbody>
</table>

The first row shows the allowed levels which the stimulus can take. The second row gives the probability of finding the stimulus at a given level. The Gaussian stimulus has a continuity of levels. The binary stimulus has two levels. The Poisson stimulus has a multiplicity of discrete levels corresponding to multiple flashes within a sampling time interval. For the binary and Poisson cases it is instructive to consider not only zero mean stimuli, but also the stimulus whose lowest level is at zero. For this latter case Table 1 shows \(\overline{x^n} = \bar{z}^{[n]} = \frac{L^n}{p^{n-1}}\) where \(z_b\) and \(z_p\) are the binary and Poisson stimuli, and

\[ z^{[n]} = z(z-1)(z-2)\ldots(z-n+1) \]  

(30)

Orthogonal polynomials for these stimuli can be obtained from the determinant in Eq. (8).
The coefficients which appeared in Eq. (29) are tabulated in Table 2 for the different stimuli:

<table>
<thead>
<tr>
<th>Power=Qu2δ</th>
<th>Q12</th>
<th>Q13</th>
<th>Q33</th>
</tr>
</thead>
<tbody>
<tr>
<td>=δ2/δ2</td>
<td>=δ2/δ2</td>
<td>=δ2/δ2</td>
<td>=δ2/δ2</td>
</tr>
<tr>
<td>Gaussian</td>
<td>σ2δ</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Binary</td>
<td>L2(Δ-δ)</td>
<td>L(Δ-2δ)</td>
<td>L2(Δ2-4Δδ-δ2)</td>
</tr>
<tr>
<td>Poisson</td>
<td>L2Δ</td>
<td>LΔ</td>
<td>L2Δ</td>
</tr>
</tbody>
</table>

In Table 2 we have introduced the time interval Δ = δ / p which is equal to the average time between the beginning of two pulses.

The equality expressed by Eqs. (27) and (29) requires the power density of the Gaussian stimulus used for obtaining the Wiener kernels to be the same as the power density of the stimulus x(t). The power density which is tabulated in Table 2 is defined as the area under the stimulus autocorrelation function:

\[ P \equiv \sum_{\tau} x(t)x(t+\tau) \delta \]  \hspace{1cm} (31)

For white noise stimuli, where \( x(t)x(t+\tau) = 0 \) for \( \tau \neq 0 \), the power density is given by:

\[ P = (\delta^2 - \bar{x}^2) \delta \]  \hspace{1cm} (32)

The coefficients \( Q_{nm} \) in Eq. (27) become uselessly complex for all non-Gaussian stimuli except the Poisson distribution in which case the result is:

\[ Q_{nm} = (L\Delta)^{m-n} = a^{m-n} \] for \( n \neq 0 \)
\[ Q_{0m} = L^{m} \Delta^{m-1} = a^{m} / \Delta \]  \hspace{1cm} (33)

where \( a, L, \) and \( \Delta \) are respectively proportional to the number of photons per flash, the number of photons per second and the average number of seconds per flash.

The relationship between the Poisson flash kernels, \( H^P \), and Wiener (Gaussian) kernels is given by Eqs. (27) and (33):

\[ H^P_{r_1 \ldots r_r}(\tau_{1}^{r_1} \ldots \tau_{r}^{r_r}) = \sum_{r_{m}} \frac{a^{m-n}}{r!} \sum_{r_{n}} H^{G}_{m}(\tau_{1}^{r_1} \ldots \tau_{k+r}^{r_r}) \left( \frac{\delta}{\Delta} \right)^{r} \]  \hspace{1cm} (34)

The first few terms of the expansion are (for some fixed power density):

\[ H^P_{1}(\tau_{1}) = H^{G}_{1}(\tau_{1}) + aH^{G}_{2}(\tau_{2}) + a^2H^{G}_{3}(\tau_{3}) + a^3 \left[ H^{G}_{4}(\tau_{1}^{4}) + \sum_{\tau_{2}} H^{G}_{4}(\tau_{1}^{4} \tau_{r_{2}}^{2}) \left( \frac{\delta}{\Delta} \right) \right] \]
\[ + a^4 \left[ H^{G}_{5}(\tau_{1}^{5}) + \sum_{\tau_{2}} \left( H^{G}_{5}(\tau_{1}^{5} \tau_{r_{2}}^{2}) + H^{G}_{5}(\tau_{1}^{5} \tau_{r_{2}}^{2}) \right) \left( \frac{\delta}{\Delta} \right) \right] + O(a^5) \]  \hspace{1cm} (35)

Suppose we knew the Wiener kernels \( H^{GP}_{1}, H^{GP}_{2}, H^{GP}_{3} \) for power density \( P' \) and we would like to estimate the Poisson kernels for power density \( P = a^2 / \Delta \). By combining Eqs. (24) and (35) the terms up to third order become:
\[ H_1^{FP}(r_1) = H_1^{GP}(r_1) + aH_2^{GP}(r_1) + a^2 H_3^{GP}(r_1^2) + (P - P') \sum_{r_2} \delta H_3^{GP}(r_1, r_2^2) \]  \hspace{1cm} (36)

CONCLUSIONS

The main points raised in this paper were:

1. An orthogonal functional expansion can be written for a general white-noise stimulus using Eqs. (8) and (9).

2. The functional expansion can be inverted to obtain the kernels by using Eqs. (12) and (14).

3. We prefer the normalization given by Eq. (14) over the conventional normalization (see Eq. 15) because it leads to simplified expansion coefficients. The relationship between Volterra and Wiener kernels is simple using our normalization (Eq. 24), whereas it is quite cumbersome using the standard normalization.

4. The expansions are simplified if diagonal elements are kept separate from off-diagonal elements.

5. General kernels measured with stimulus \( w(t) \) can be written as an expansion of kernels measured with stimulus \( x(t) \). Eq. (18) is the general formula, and Eqs. (19) and (20) show examples for lower-order kernels.

6. An important distinction was made between exclusive summations in which the summations do not include the diagonal elements, and inclusive summation in which the diagonal elements are included. The general formalism is much simpler for exclusive summation. Inclusive summation, however, is necessary for filtered stimuli that are not perfectly white where the extent of the diagonal is not clearly defined.

7. An elegantly simple formula (Eq. 24) was presented for relating Gaussian kernels obtained with stimuli of different power densities. The form of the expansion is identical to a Taylor's series. The summations are inclusive. A formula that includes luminance changes was also presented (Eq. 25). Since luminance and power density are the only parameters that are needed to characterize a Gaussian signal, Eq. (25) is a general expression relating kernels obtained using two arbitrary Gaussians.

8. The inclusive expansion of a general kernel in terms of Gaussian kernels of the same power density was presented in Eq. (27) with some low-order examples given in Eq. (29).

9. The last portion of the paper focuses on Gaussian, binary and Poisson stimuli. Table 1 gives the low-order moments of these stimuli and Table 2 gives the low-order generalized cumulants of the stimuli. The generalized cumulants are the expansion coefficients relating the kernels.
10. Eqs. (34–36) are the final equations and possibly the most useful. They relate Poisson kernels to Gaussian kernels. This connection is especially important because the Poisson kernels are obtained with a stimulus consisting of abrupt random flashes. For low flash rates, the Poisson stimulus is equivalent to the transient single-flash commonly used in most physiology experiments. Thus Eqs. (34) and (36) establish a connection between white-noise stimuli and classical stimuli.

REFERENCES


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