Visual multipoles and the assessment of visual sensitivity to displayed images.
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ABSTRACT

The contrast sensitivity function (CSF) is widely used to specify the sensitivity of the visual system. Each point of
the CSF specifies the amount of contrast needed to detect a sinusoidal grating of a given spatial frequency. This paper
describes a set of five mathematically related visual patterns, called "multipoles," that should replace the CSF for measuring
visual performance. The five patterns (ramp, edge, line, dipole and quadrupole) are localized in space rather than being spread
out as sinusoidal gratings. The multipole sensitivity of the visual system provides an alternative characterization that
complements the CSF in addition to offering several advantages. This paper provides an overview of the properties and uses
of the multipole stimuli. This paper is largely a summary of several unpublished manuscripts with excerpts from them.
Derivations and full references are omitted here. Please write me if you would like the full manuscripts.

1. DEFINITIONS OF MULTIPOLES (from Klein 1)

1.1 Contrast units.

The luminance of a general one-dimensional visual pattern can be written as:

\[ L(x) = L_0 (1 + P(x)) \]  

where \( L_0 \) is the background luminance and \( P(x) \) represents the fractional luminance change as a function of position of the
pattern. \( P(x) = (L(x) - L_0) / L_0 \). \( P(x) \) has units of contrast rather than luminance because the background luminance, \( L_0 \), is
factored out. Contrast units are used throughout because they are appropriate for photopic vision where Weber's law holds so
that contrast thresholds are approximately independent of luminance.

1.2 Idealized Multipoles.

The point multipole, \( P_m(x) \), is defined as the mth derivative of a delta function, \( \delta(x) \), where \( m \) designates the multipole
order.

\[ P_m(x) = M_m \frac{d^m}{dx^m} \delta(x) \]  

The delta function (proportional to the \( m = 0 \) line multipole) for the continuous case has an unusual definition. It is zero
everywhere except at \( x = 0 \), where it is infinite. The area under \( \delta(x) \) is unity:

\[ \text{Area} = \int \delta(x) \, dx = 1 \]  

In principle the line multipole has zero width. However, in applications to visual stimuli the line width can be finite but
smaller than the eye's line spread function. Multipoles that are wider than the eye's line spread function will be called blurred
multipoles (discussed later).

The constant \( M_m \) in Eq. 2 is the multipole moment that measures the strength of the mth multipole. The units of the
multipole moment can be derived by examining the units of all the other factors in Eq. 2. \( P_m(x) \) has units of \% contrast as
seen from Eq. 1. The mth derivative, \( \frac{d^m}{dx^m} \), has units of \% per unit length. \( M_m \) will be in min of arc throughout this paper. The
delta function \( \delta(x) \) has units of \% per unit length. In order to balance the units in Eq. 2, \( M_m \) must have units of \% per unit length.
Thus, as expected, an edge (\( m = 1 \)) has units of \% and a line (\( m = 0 \)) has units of \% per min of arc (contrast times width).
The ramp (\( m = 2 \)) has units of \% per min of arc, which is reasonable since the ramp strength is specified by the contrast gradient. The
dipole (\( m = 1 \)) and quadrupole (\( m = 2 \)) have units of \% per min of arc and \% per min of arc.

Negative values of \( m \) in Eq. 2 stand for integration rather than differentiation. Thus the edge multipole for \( m = 1 \) is
given by:

\[ P_{-1}(x) = \int_0^x \delta(x) \, dx \]

\[ = -M_{-1}/2 \quad \text{if } x < 0 \quad \text{or } = M_{1}/2 \quad \text{if } x > 0 \]

It is important to note that this definition of edge strength as given by the edge multipole moment differs by a factor of 2
from the Michelson definition of contrast. Thus an edge with \( M_1 = 1 \% \) would have a Michelson contrast of .5%.

The ramp multipole is given by a double integral of the delta function. By invoking symmetry as was done for the
edge one can write the ramp stimulus as:
\[ P_2(x) = M_2 \int_0^x \int_0^x \delta(x') \, dx' \, dx \] 
\[ = M_2 \frac{1}{2} h/2 \]

1.3 Discrete multipoles.

This paper focuses on physically realizable stimuli. Since computer generated video displays are based on pixels, finite separations rather than continuous positions, a one-dimensional video pattern must be represented as a finite string of numbers, each one representing the contrast of a single raster line on the screen. For example, a line whose width is one pixel can be written as
\[ \delta(l) = (\ldots 0 \quad 0 \quad 0 \quad 1/\Delta \quad 0 \quad 0 \quad 0 \ldots) \] 
\[ (4) \]
where \( \Delta \) is the angular distance between pixels and \( 1/\Delta \) is the value of the discrete delta function at \( t=0 \). The index, \( l \), is the pixel label. The discrete delta function is defined (Eq. 4) to be zero everywhere except at the origin, where its value is \( 1/\Delta \) so that the area under the curve equals unity.

1.4 Moments of general multipoles.

A formula is needed for calculating the strength of a general multipole. For discrete patterns and \( m \leq 0 \), Eq. 2 can be inverted to give the multipole moment:
\[ M_m = \Delta^{(m+1)} \, (-1)^m \frac{m!}{m!} \sum_l \Delta^{|m|} P_m(\Delta) \] 
\[ (5) \]
For \( m=0 \) this equation becomes
\[ M_0 = \sum_l P_0(\Delta) \] (or \( M_0 = \int_0^\infty P_0(\Delta) \, d\Delta \) for the continuous case) since the line moment is given by the area under the line. Consider, for example, the case of a single line removed from a uniform background composed of equally spaced pixels with separations of \( \Delta \) \( \min \). The resulting black line would have a multipole moment of -100 \( \Delta \) \( \% \min \), independent of line luminance, since the contrast of the line is -100\%. For \( m=1 \), Eq. 5 becomes
\[ M_1 = \Delta^2 \frac{\Delta}{\Delta^0} \] (or \( M_1 = \int_0^\infty P_1(\Delta) \, d\Delta \)). A dipole can be generated by shifting a single line by \( \Delta/2 \) where \( \Delta \) is the pixel spacing. This dipole is equivalent to the original uniform background plus a -100\% contrast black line at \( \Delta/2 \) (since the \( \Delta \)=0 line was shifted by 1/2 pixel). The dipole moment is therefore
\[ M_1 = 50 \Delta^2 \% \min ^2 \] . As a final example consider a quadrupole that consist of four adjacent pixels with spacing of \( \Delta \), and with contrasts of -100\%, +100\%, +100\%, and -100\% at \( i=0, 1, 2, \) and 3. Eq. 5 implies that this stimulus has a quadrupole moment of
\[ 100 \Delta^2 \% \min ^2 \]

1.5 Rectangular Blur and sequentially ordered multipoles.

This section examines the effect of blur on multipoles. A blurred multipole can be generated by starting with a Gaussian (\( \exp(-x^2/2\alpha^2) \)), Cauchy (\( 1/(x^2+\alpha^2) \)), or rectangular function as the zeroth order multipole instead of a delta function in Eq. 2. Rectangular blur is based on the following zeroth order multipole: \( R(x) = 1/d \) for \( 0 \leq x < d \), and \( R(x) = 0 \) outside this interval. This function is normalized to have unity area. The full family of blurred multipoles can be obtained by taking derivatives or integrals of the zeroth order multipole.

Rectangular blur is especially interesting because it clarifies how a multipole of order \( m \) is related to a multipole of order \( m-1 \). A rectangular blurred line, \( R(x) \), is a bar that is identical to two opposite-polarity closely spaced edges (an increase in luminance followed by a decrease in luminance). The first derivative of a rectangular blurred line is a blurred dipole, a pair of opposite-polarity delta functions:
\[ D[R(x)] = (6 \times (\delta(x) \cdot \delta(x-d))/d \]
where \( D \) is defined to represent the derivative operator for the continuous case or the difference operator for the discrete pixel case. An \( m \)th order multipole with rectangular blur, \( P_m(x) \), can be written according to Eq. 2 as:
\[ P_m(x) = M_m \cdot b^{m-1} D[R(x)] \]
\[ = M_m \cdot (D^{m-1} \delta(x)) \cdot (D^{m-1} \delta(x-d))/d \]
\[ = P_{m-1}(x) \cdot P_{m-1}(x-d) \]
where \( P_{m-1}(x) \) is a point multipole (unblurred) whose order is \( m-1 \), and whose multipole moment, \( M_{m-1} \), is
Thus, two adjacent opposite-polarity multipoles of order m-1, moment M, and separation d, produce a multipole of order m and moment M.d.

\[ M_{m-1} = M_m / d. \]

(6)

Blurr has no effect on the multipole moment. In the Fourier domain, \( M_m \) is the spatial frequency power density of the multipole near zero spatial frequency. Thus a line multipole of 100% contrast and 2 sec in width has the same moment (3.3 \%min) as a line of 10\% contrast and 20 sec in width, and even the same moment (but not the same visibility) as a line of .1\% contrast and 2,000 sec in width.

2. GENERATING MULTIPOLES ON VIDEO TERMINALS.

This being a conference concerned with video displays we now discuss how multipoles are generated on video terminals. The edge and ramp multipoles are straightforward to generate since they are familiar stimuli. The only requirement is that the stimulus generator must be capable of fine gradations of grey levels. Suppose, for example, that the maximum luminance of the display is 300 cd/m² and it is desired to measure the visibility of an edge on a 30 cd/m² background. The edge threshold of .7\% corresponds to \( \Delta L = .21 \text{ cd/m²} \), which is .07% of the maximum luminance. It requires more than 10 bits of luminance control to achieve this small increment. In order to simultaneously have some areas of the display at high luminance and other areas at low luminance with near-threshold contrasts the stimulus generator should have a 12 bit capability.

Generation of the line multipole requires a discussion of how the line spread function of the display affects the multipole moment. Suppose that on the display the separation between scan lines is .1 min and the full blur width of each scan line is .05 min. The finite width of each scan line results from the fact that it is impossible to have perfect focus on a real display. If the luminance of one of the scan lines of the uniform field were cut in half, then a thin dark line would be present. The strength of that line can be most easily calculated by first blurring lines of the display so that instead of having a width of .05 min each line has a uniform luminance across .1 min so that the pixel width matches the pixel separation. In that case the luminance would be perfectly uniform across the entire display. The strength of the one pixel dark line would be the product of its 50\% contrast times its .1 min width, for a line moment of 5\% min. As was mentioned in the preceding section, blurring the line does not alter the value of the line moment. Thus the original display with thinner lines also had a line strength of 5\% min. Furthermore, the display as viewed by a visual system with the relatively large blur of 1 min would still have a line strength of 5\% min. As the blur extent increases, the multipole moment stays the same, but the visibility of the multipole decreases because the high spatial frequency content of the multipole decreases, as will be discussed in the next section.

Another factor of practical importance is the orientation of the line multipole. So far we have only discussed lines that are oriented in the same direction as the raster. The very high temporal frequency properties of the video circuitry are not relevant since the luminance of the line is constant across the scan. However, for lines that are perpendicular to the scan direction there are grave calibration problems. Consider, for example, a 60 Hz display with 312 x 312 pixels. The duration of each pixel is only 63.4 nsec. It is unlikely that the intensity of the pixel can reach its final value during that short time interval. Consequently, a 1 pixel line perpendicular to the scan direction would have a lower contrast and a lower multipole moment than a line with the same specifications that is oriented along the scan direction. Furthermore, a line perpendicular to the scan that is two pixels wide would have a strength more than twice that of a single line since the intensity would get closer to its asymptotic value. It is for these reasons that whenever using local multipoles it is best to use stimuli that are oriented parallel to the scan direction.

From the considerations of the preceding paragraph it might be surmised that as long as the lines are oriented along the scan direction interactions between adjacent pixels are minimal. This need not be the case. For a 60 Hz display with 512 lines, the time between lines is about 32 \( \mu \)sec. The electronics should be fast enough to recover during this relatively long interval. However, even though the electronics might recover fully, the phosphors might not. Thus if the screen blurs so that there is some phosphor overlap between adjacent scans, then one would expect to find nonlinear interactions between adjacent pixels. To illustrate this interaction consider a dipole that is generated by the pixel shifting method. Start with a field that consists of alternating white and black pixels with the spacing between white pixels less than 1 min (a spatial frequency greater than 60 c/deg) so that the field looks uniform. Now make one of the white pixels black and the following black pixel white. The beauty of this scheme is that even if the display monitor were horribly nonlinear the strength of the new black line should be exactly equal and opposite to the strength of the new white line thereby creating a balanced dipole. Suppose the following three conditions held: 1. The phosphor is operating near its threshold so that luminance is an accelerating function of applied voltage (commonly found in displays). 2. The phosphor has a "memory" that lasts longer than the 32 \( \mu \)sec between scan lines. 3. The original display with alternating pixels has minimal pixel overlap, but adjacent pixels have significant overlap. In this case the scan over the white line of the dipole would overlap
some of the same phosphors as the previous scan and because of the accelerating luminance nonlinearity the brightness would be greater than expected. The severity of this problem can be assessed using the following psychophysical technique. On one half of the screen one places the original uniform field consisting of an alternating one-on one-off display. On the other half of the screen one places a display consisting of alternating two-on two-off pixels. The viewing distance must be sufficiently large that the latter field appears to be uniform. Any nonlinearity of the sort we are discussing would show up as a luminance imbalance between the two halves. Because of the high edge sensitivity of the visual system, one can pick up a luminance step of less than 1%, which is about as good as can be done with a photometer. If a photometer is used one must take steps to remove possible artifacts. The photometer should not scan different portions of the display. Rather, the display should be switched from one pattern to the other. Also a big spot size should be used to minimize edge effects.

3. VISIBILITY OF MULTIPLETS (from Klein and Eastlake)

3.1 Visibility of unblurred multiplets.

Up until now we have discussed multiplets in mathematical terms. We now discuss their visibility as seen by the human visual system. The second row of Table 1 gives some typical values of multipole thresholds for luminances of about 100 cd/m² and for a threshold criterion set at D=1. The bottom row shows sample stimuli where the middle of the luminance range has been set to 5 units.

<table>
<thead>
<tr>
<th>multipole</th>
<th>ramp (m=2)</th>
<th>edge (m=1)</th>
<th>line (m=0)</th>
<th>dipole (m=1)</th>
<th>quadrupole (m=2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>threshold</td>
<td>0.7% /min</td>
<td>7.7%</td>
<td>1.3% /min</td>
<td>1.7% /min²</td>
<td>1.0% /min³</td>
</tr>
<tr>
<td>samples</td>
<td>(987656789)</td>
<td>(444466666)</td>
<td>(555595555)</td>
<td>(555591555)</td>
<td>(5555991555)</td>
</tr>
</tbody>
</table>

Table 1

Two familiar stimuli are the edge and line which have thresholds of about 7% (twice the Michelson value) and 1.3% /min. This value for the edge threshold is for the case where no temporal transients are present. A stimulus with abrupt onsets or offsets or with flicker would exceed the edge threshold by more than a factor of two. The higher order multiplets are less affected by temporal factors. The line threshold of 1.3% /min means that a line of 100% contrast and 0.015 sec (.78 sec) in width would be at threshold.

The threshold for dipole and quadrupole are less familiar. Their visibility will be discussed next in the context of the dipole being two opposite polarity lines and the quadrupole being two opposite polarity dipoles. The definition of ramp and edge thresholds can be ambiguous. The multipole formalism removes any ambiguity. The edge threshold is based on M = A/m = (T_max - T_min)/L, which is twice the Michelson contrast. The ramp multipole is also based on the difference in gradient (T_max - T_min)/L, rather than the maximum gradient.

3.2 Visibility of opposite-polarity multipole pairs vs. separation. Generalized Riccio's Law.

A previous section showed that a closely spaced pair of opposite polarity multiplets of order m, is the same as a multipole of order m+1, with rectangular blur. Consider first the rectangular blur of a line. As indicated in Table 1 the threshold for an unblurred line (a single pixel) is about 1.3% /min and for an edge it is about 7%. The adjacent figure shows the visibility of a bar (blurred line) whose width is plotted on the abscissa. The vertical axis is the sensitivity, which is the reciprocal of the contrast of one of the edges comprising the bar when the bar is at threshold. The data shows that for wide separations the sensitivity is fixed at about 1.4%⁻¹, corresponding to an edge contrast of .7%. For bar widths less than 1 min, the sensitivity is seen to be linearly proportional to the width so that the product of contrast times width is a constant. This is called the Riccio funnelling region. For widths within this Riccio's regime, the bar is detected not on the strength of the edge multipole but rather on the strength of the line multipole. The line multipole, remember, has units of /min, so that a constant line multipole would be represented by a line with unity slope on this log-log plot. On a linear plot the slope of the line would equal the line multipole strength. On a log-log plot the strength of the line can be calculated by choosing a point at the left of the plot and multiplying the width and the contrast of that point. For the data shown, the line threshold is about 1.2% /min.

The size of Riccio's integration area can be determined by drawing two straight lines: a horizontal line based on the edge threshold corresponding to the data on the right of the plot, and a line of unity slope based on the line threshold, corresponding to the data on the left of the plot. The horizontal location where these two lines intersect is the extent of
Ricco's area. For the data shown in the figure Ricco's width is about 1.8 min. In the idealized case of a detection mechanism with a rectangular profile, a bar whose width was smaller than the Ricco's width would be detected by the total area (full summation) under the bar. For bars wider than the critical width, summation is no longer complete and the bar would be detected based on the contrast rather than the integrated area. The actual data depart from the idealized case of a detector with a rectangular profile. For widths near the critical Ricco width, the sensitivity is greater than what would be expected from detection being based on the visibility of a single edge. The two opposite polarity edges must summate. This overshoot can be achieved if the underlying mechanism has a receptive field in which a central excitatory zone is flanked by a pair of inhibitory zones.

Mathematically, the connection between the line threshold, L, the edge threshold, E, and Ricco's extent, s, is easy to establish. Consider a bar in which the contrast of the two opposite polarity edges is at the edge threshold, E. If the bar separation is greater than s, the bar threshold would be determined by the edge strength. If the bar width is less than s, the bar threshold would be determined by the line strength, L, given by the product of contrast times the width. When the width equals s, the threshold line strength must equal the threshold edge strength times the width:

\[ L = E \times s \]  

(7)

Ricco's integration region is thus:

\[ s (\text{min}) = L (\% \text{min}) / E (\%) \]  

(8)

Thus an easy way to measure the size of Ricco's integration zone is to measure the line and edge thresholds and to take their quotient. This procedure is easier than the typical method of measuring the entire curve as was done in the above figure.

So far we have only discussed the line-edge transition. The same discussion applies to every adjacent-order multipole pair. The generalized Ricco's integration zone can be written as:

\[ s_m (\text{min}) = M_m (\% \text{min}^{m+1}) / M_{m-1} (\% \text{min}^{m}). \]  

(9)

Based on Table 1 and Eq. 9, the ramp-edge, edge-line, line-dipole, and dipole-quadrupole transitions are at separations of 10, 1.9, 1.3, and .6 min. Thus for two opposite polarity ramps as shown in panel a, for widths greater than s = 10 min, threshold is determined by the luminance gradient.

For smaller widths, threshold is determined by the total change in luminance (the edge-multipole). The case of the bar (opposite polarity edges) in panel b needs no further discussion since that was the focus of this section and is the classical Ricco's case.

For the pair of opposite polarity lines in panel c and for widths larger than s = 1.3 min, threshold is based on the visibility of a single line. For smaller widths the dipole moment determines threshold. In the present context we view s as the integration zone for a dipole. It is smaller than the integration zone for a line, as might be expected since the dipole has higher spatial frequencies with a correspondingly smaller spatial extent. An alternate approach is to view s as a cancellation region. Thus whereas s is the extent over which a bar (multipole same-polarity lines) will summate, s is the region within which two opposite-polarity lines will cancel. Also relevant to the issue of integration zones and cancellation zones is data on the summation of a pair of same-polarity multipoles (see Klein and Eastlake for data on thresholds of two same-polarity multipoles as a function of separation). The dipole threshold also provides an estimate of the maximum gradient of the receptive field. Panel d shows how a quadrupole consists of a pair of opposite polarity dipoles. Klein and Eastlake provide further details.

4. RELATIONSHIP BETWEEN MULTIPOLe SENSITIVITY AND SINUSOID SENSITIVITY (from Klein)

4.1. Bandwidth of mechanisms.

In order to connect multipole sensitivity to sinusoid sensitivity a new definition of mechanism bandwidth is needed. For any localized function \( F(t) \), the bandwidth, W, will be defined as follows:

\[ W = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{F}(t) \, dt \]  

(10)

\[ = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{F}(t) \, d\log(t) \]  

(11)

where a function with a hat has been normalized to have a unity maximum as given by: \( \hat{F}(t) = F(t) / \text{Max} \{ F(t) \} \). The normalization factor \((2\pi)^{-1/2}\) was chosen so that if \( F(t) \) is the positive frequency portion of a Gabor function.
\[ F(f) = \exp\left(-\frac{(f-f_0)^2}{2\sigma^2}\right) \]

with standard deviation, \( \gamma \), then the bandwidth (Eq. 10) is calculated to be \( \gamma f_0 \) in the limit of \( \gamma << f \). This result is a consequence of the Gaussian normalization: \( \int_{-\infty}^{\infty} F(f) \, df = (2\pi)^{1/2} \). Thus for functions with narrow tuning, this bandwidth definition is approximately the standard deviation (calculated with a linear frequency axis and a linear sensitivity axis) of the mechanism divided by the peak spatial frequency of the mechanism. The bandwidth definition of Eq. 10 is even more persuasive with a logarithmic frequency axis (and linear sensitivity) since the ratio \( \gamma / f_0 \) on linear frequency axis is the standard deviation of the function F(f) when plotted on a logarithmic axis (natural log).

The definition of bandwidth given by Eq. 10 has a natural interpretation in terms of the number of cycles in the receptive field of the mechanism. For the antisymmetric Gabor case, the spatial receptive field is given by:

\[ \exp(-\gamma f_0^2 / 2) \sin(f_0 x) \]

where \( x \) is in deg and \( f_0 \) and \( f_0 \) are in rad/deg. The envelope has a standard deviation of 1/\( f_0 \) and the period of one cycle is 2\( f_0 \). The number of periods within \( \pm 1.5 \) standard deviations is about 2\( \pi / 3 \) \( f_0 / f_0 \) = 2.1/W since the \( \pm 1.5 \) points are where the Gaussian envelope fails to about 1/3 of its peak value (exp(-1.5^2/2) = 1/3). Thus, the number of cycles in the receptive field is about twice the inverse bandwidth. Equivalently, 1/W is approximately the number of half-cycles in the mechanism's receptive field. The bandwidth W is also approximately 3 times the number of octaves between the half-maximum points.

4.1.1 Examples using Cauchy and Gaussian functions. The bandwidth of the \( n \)th order Cauchy function whose spatial frequency tuning is

\[ F_n(f) = f^p \exp(-\gamma f) \]

can be calculated from the definition of the gamma function (factorial function):

\[ W_n(f) = (2\pi)^{1/2} n^{n-1} \exp(n) \]

An approximation that is good to 1% for \( n > 8 \) can be derived from Stirling's gamma function approximation:

\[ W_n(f) = n^{n/2} \]

The \( n \)th frequency moment of a Cauchy function (the \( n \)th order function) corresponds to the spatial domain to the \( n \)th spatial derivative of a symmetric or antisymmetric pair of blur functions given by \( 1/(\sigma^2 + x^2) \) and \( 1/(\sigma^2 + x^2) \). These functions are especially useful for vision experiments and vision modelling because they are Hilbert pairs with the same spatial frequency tuning. In addition, the normal contrast sensitivity function is well fit by a Cauchy function since log sensitivity is linearly related to spatial frequency for spatial frequencies above the peak:

\[ CSF(f) = A B \exp(-p f/h) \]

where \( p = 5, h = 9 \) rad/min (= 8.6 c/deg), and \( A=500 \) are parameters for a normal CSF. This value of \( h \) corresponds to a CSF fall-off of 1.01 dB per c/deg at high spatial frequencies, an easily memorized value.

The bandwidths of Gaussian functions are also easily calculated using the new formula. The \( n \)th derivative of a Gaussian with a standard deviation of \( 1/\sqrt{2} \) has a spatial frequency tuning proportional to:

\[ F_n(f) = f^p \exp(-\gamma f^2) \]

The bandwidth of these functions can be exactly related to Cauchy bandwidths according to:

\[ W_n(f) = 1/2 \ W_n(f/2) \]

For example, \( W_2(f) = 1/2 \ W_2(f/2) = 1/2 \cdot 737 \approx 368 \). Stirling's approximation for the Gaussian case gives:

\[ W_n(f) = (2n)^{1/2} \]

Thus an eighth derivative of a Gaussian has a bandwidth of \( W = (2x)^{1/2} \) = .25 and two cycles, \((2W)^{-1}\), in the receptive field. The normalization factor \((2\pi)^{1/2}\) in Eq. 10 was needed to arrive at the simple relationship shown in Eqs. 14 and 17.

4.2 Relationship between multipole sensitivity and sinusoid sensitivity

In order to connect multipole sensitivity to sinusoid sensitivity we must make several assumptions:

1. The multipoles are detected by the same mechanisms that detect the sinusoids. Without this assumption there can be no connection between multipole and sinusoid sensitivity.
2. The multipoles are detected by mechanisms with the optimal phase. Thus even order multipoles are detected by symmetric mechanisms and odd order multipoles are detected by antisymmetric mechanisms. The consequences of relaxing this assumption are discussed later.

3. For this paper we also assume that a pattern is detected when the single most sensitive mechanism reaches its threshold. Klein\(^1\) considers the case where a pooled group of mechanisms determine threshold such as by probability summation.

4. The mechanisms all have the same bandwidth but differing sizes. This assumption of equal bandwidth is not necessary but it simplifies the discussion.

Let us start with the constraint based on an edge sensitivity of .7%. This multipole sensitivity places an upper limit on the sensitivity of all the underlying detection mechanisms. The upper limit on mechanism sensitivity in turn places an upper limit on the sinusoid sensitivity. These various ingredients combine to produce the main equation of Klein\(^1\):

\[
U_m(f) = \frac{(\pi/2)^{1/2} W(m+1)M_m}{(W(n+1))^{1/2} W(n+1)M_m}
\]

where \(U_m(f)\) is an upper limit constraint line that is placed on the contrast sensitivity function because of the sensitivity of the visual system to the \(m\)th order multipole, \(M_m\). The factor \(W(m+1)\) is the bandwidth (Eq. 10) of the \((m+1)\)th moment of the underlying mechanisms. If the underlying mechanisms are assumed to be \(n\)th order Cauchy functions then the bandwidth term becomes \(W(n+1)\) since the \(n\)th moment and \(n\)th order are additive because of the form of the Cauchy function (Eq. 13). Some examples are useful to help clarify Eq. 18. For the edge and line multipoles Eq. 18 becomes

\[
U_1(f) = \frac{(1.253/W(0))}{M_1}
\]

\[
U_0(f) = \frac{(1.253/W(n))}{(M_0)}
\]

where \(1.253 = (\pi/2)^{1/2}\). If the edge and line thresholds have the value given in Table 1 (\(M_1 = .7\%\) and \(M_0 = 1.3\%\)) and the detection mechanism is a third order Cauchy function (\(W(0)\) and \(W(n)\) become .594 and .511 for \(n = 3\) then Eqs. 19 and 20 become:

\[
U_1(f) = 3.01\%^{-1}
\]

\[
U_0(f) = 1.89\%/\text{rad/min}
\]

and \(f\) is in rad/min. The upper limits \(U\) can be converted to more familiar terms by converting the percent to decimal notation and by expressing \(f\) in c/deg:

\[
U_1(f) = 301
\]

\[
U_0(f) = 1805/f
\]

Note that the edge threshold, \(M_1\), forces an upper limit constraint that is a constant. The CSF must be less than 301.

If this condition were violated, so that at a spatial frequency of \(f\) the CSF were greater than 301, then the mechanism that detected the sinusoid of frequency \(f\), would have sufficient sensitivity to detect an edge whose contrast was less than .7%.

Our finding that the edge contrast was .7%, places an upper limit on sinusoid sensitivity. Note that this upper limit is inversely proportional to the bandwidth of the underlying mechanisms. A very narrow bandwidth (many lobes in the receptive field) means that the visual system will be very sensitive to sinusoids since the multiple lobes of the stimulus will match the many lobes of the mechanism. The multipoles, on the other hand, are only sensitive to a single lobe of the stimulus since they are tiny in space, being smaller than the blur function.

Eq. 22 implies that the line threshold of 1.3% places an upper limit on the CSF that is inversely proportional to \(f\). This constraint line has a slope of -1 on log-log axes. Thus at spatial frequencies of 10 and 20 c/deg the CSF must be below 180 and 90 respectively. Table 2 summarizes the constraints on the CSF based on the 5 multipole thresholds specified in Table 1, on the assumption that the underlying detection mechanisms are \(n=3\) Cauchy functions.

<table>
<thead>
<tr>
<th>Order</th>
<th>Moment</th>
<th>Bandwidth</th>
<th>(U_1(f))</th>
<th>(U_0(f))</th>
<th>(c/\text{deg})</th>
<th>(20\ c/\text{deg})</th>
</tr>
</thead>
<tbody>
<tr>
<td>ramp</td>
<td>0.7%</td>
<td>.07%</td>
<td>1.7</td>
<td>6160 f</td>
<td>645 f</td>
<td>6,450</td>
</tr>
<tr>
<td>edge</td>
<td>.7 %</td>
<td>2.11</td>
<td>301</td>
<td>301</td>
<td>301</td>
<td>301</td>
</tr>
<tr>
<td>line</td>
<td>1.3%</td>
<td>2.46</td>
<td>189/f</td>
<td>1805/f</td>
<td>180</td>
<td>90</td>
</tr>
<tr>
<td>dipole</td>
<td>1.7%</td>
<td>2.76</td>
<td>162/f(^2)</td>
<td>14,800/f(^2)</td>
<td>148</td>
<td>37</td>
</tr>
<tr>
<td>quadrupole</td>
<td>1.0%</td>
<td>3.08</td>
<td>308/f(^3)</td>
<td>268,000/f(^3)</td>
<td>268</td>
<td>33.5</td>
</tr>
</tbody>
</table>

**Table 2**

Figure 3 shows the upper limit constraint lines on log-log coordinates. Notice that the ramp constraint is dominant at low spatial frequencies, the edge constraint is relevant near the CSF peak. The higher order multipoles place constraints on the CSF at increasing spatial frequencies.
5. CONNECTION BETWEEN MULTIPOLES AND SPATIAL JUDGEMENTS

5.1 Relationship between multipoles and vernier acuity (from Klein, Casson and Carney)

When a line or dipole test pattern is added to an edge or line pedestal respectively, a spatial shift in the pedestal is produced. This connection between multipole positions and product shifts can be understood in terms of the Taylor's series expansion. For a very small offset, \( \Delta \), the shifted pattern can be written using the first two terms of a Taylor's expansion:

\[
P(x + \Delta) = P(x) + \Delta P'(x)
\]

(23)

We would now like to rewrite Eq. 23 to express the edge pattern, \( P_E \), in terms of the product of the strength of the edge, \( M_E \), times an edge of unit strength, \( U_E(x) \):

\[
M_E U_E(x + \Delta) = M_E U_E(x) + \Delta M_E U_E'(x)
\]

(24)

the rightmost term can be written as \( T \_L U_L(x) \) since a unit strength line is the derivative of a unit strength edge (\( U_L = U_E \)) and where \( T_L = M_E \) is the strength of the added test pattern. Thus the size of the vernier offset, \( \Delta \), is given by the strength of the test pattern, \( T_L \), divided by the strength of the pedestal, \( M_E \), or \( \Delta = T_L / M_E \). This same relationship holds for an nth order multipole which can be expressed by removing the subscripts L and E:

\[
\Delta = T / M
\]

(25)

where \( M = M_{n-1} \) is the pedestal and \( T = M_L \) is the test. It is useful to illustrate Eq. 25 with a quantitative example. If a threshold line, whose strength is \( T_L = M_L = 2 \) is added to an edge pedestal of strength \( M_{n-1} = 50 \)%, then the edge is shifted by an amount \( \Delta = 2.5 \) sec.

The question of whether a stimulus leads to hyperacuity thresholds is determined by the extent to which the test pattern is masked by the pedestal. The amount of masking can be quantified by assuming that the tvi curve has the form of a straight line on log-log coordinates, which is a power function on linear coordinates:

\[
T/T_0 = (M/M_0)^n
\]

(26)

where \( T \) is the threshold test strength for a pedestal strength of \( M \), and \( T_0 \) is the threshold test strength for a pedestal at its own threshold, \( M_0 \). An important relationship between \( T_0 \) and \( M_0 \) is given by Eq. 8, the expression for Ricco's integration extent:

\[
s = T_0 / M_0
\]

(27)

where \( s \) is the size of Ricco's integration zone for the test pattern in terms of the threshold strength, \( M_0 \), of two opposite polarity pedestal stimuli. By combining Eqs. 25, 26 and 27 we arrive at an expression for the ratio of the vernier threshold to the Ricco extent in terms of the pedestal strength:

\[
\Delta t = (M/M_0)^{0.5}\Delta t
\]

(28)

This equation is elegant in that the strength of the test pattern is eliminated. Eq. 28 implies that when the pedestal is near its threshold the vernier offset is equal to the Ricco's summation extent. Furthermore, if the masking exponent is \( n = 3 \) as was found by Klein, Casson and Carney and Carney and Klein then when the pedestal is 20 times its threshold, the vernier offset is 1/50 of the Ricco extent (a hyperacuity). This connection between the hyperacuity threshold and the Ricco's extent is not exact because Eq. 26 is an approximation. The true tvi curve isn't a simple power function, but it does have the approximate shape given by Eq. 26 and we do believe that Eq. 28 is valid to within a factor of 2. This insight into hyperacuity thresholds is as quantitatively accurate as any previous model, and it involved no assumptions about the properties of underlying mechanisms. One mystery is that we found the tvi exponent to be about \( n = 3 \) which is a bit lower than the tvi slope found by others for contrast discrimination. Further research is needed to improve our understanding of the factors contributing to the shape of the tvi masking function. For some tasks Weber's law applies with \( n = 1 \), for contrast discrimination \( n = 0.5 \), and for other tasks, such as hyperacuity and resolution, the slope is much flatter.

The above analysis applies not only to vernier acuity but also to bisection thresholds for optimum separations. When the separations between the three lines are about 1.5 min, the bisection threshold has the same dependence on the contrast of the stimulus as in the vernier case. A detailed analysis of 3-line bisection thresholds and 2-line interval discrimination thresholds is given by Levi and Klein for general separations, contrasts and also for blur. A very simple Weber's Law analysis is able to account for thresholds under a very wide range of conditions.
5.2. Relationship between multipoles and resolution (from Carney and Klein, 1989).

An analysis similar to that of vernier acuity can be done for resolution. Hyperacuities such as vernier involve the detection of the shift in position of the pedestal. A small position shift can be produced by adding the first derivative of the pattern (a multipole whose order is one higher than the pedestal). A blur threshold, on the other hand, involves the detection of a change in the second derivative of the pedestal. Carney and Klein show that a Gaussian whose variance is \( \sigma^2 \) can be approximated by the following Taylor's expansion:

\[
G(y + \varepsilon, x) = G(y, x) + \varepsilon/2 \, G'(y, x)
\]  

This equation which was derived for Gaussians, applies equally well to any localized pattern such as a multipole. Thus a quadrupole added to a line gives a tilt with an increased variance. The test-pedestal approach can be applied to the resolution of lines and edges just as it was applied to the hyperacuity of lines and edges. An analysis similar to that given for Eq. 25 leads to a relationship for the threshold for detecting a variance change in terms of the threshold of a multipole test pattern, \( T \):

\[
\varepsilon = \frac{2T}{M}
\]  

where \( M = M_{q, 2} \) is the pedestal moment and \( T = M_{q} \) is the test moment. One of our surprising findings is that when expressed in terms of multipole thresholds, resolution thresholds are about two to three times lower than hyperacuity thresholds. This result is backwards from the usual statement that hyperacuity thresholds are about 5 to 10 times smaller than resolution thresholds. The problem with the usual description is that it only involves the spatial extent of the threshold while ignoring the contrast of the stimulus. When both spatial extent and contrast are included, the relative magnitude of the acuities are reversed.

The tvi curve for resolution threshold of a line involves a plot of quadrupole test strength vs. line pedestal strength. Resolution threshold for an edge involves a plot of dipole test threshold vs. edge pedestal strength. The tvi slope in both of these cases has a very low masking exponent of about \( n = 3 \), very similar to what we found for hyperacuity. The main difference is that at low contrasts there was a dipper function similar to that of contrast discrimination that was not found in the vernier case. The dipper function at low contrasts makes the resolution thresholds lower than the vernier thresholds. Because of the square root relationship between standard deviation and variance (and ignoring the dipper), Eq. 30 becomes

\[
\sigma/s = (M/M_0)^{(n-1)/2}
\]

for the threshold blur size, \( \sigma \), where \( s \) is still the Ricco size as in Eq. 28. See Carney and Klein for details.

6. CLINICAL IMPLICATIONS

An important use of multipoles is as a battery of clinical tests for assessing visual performance. This section provides a list of reasons for why we believe the battery of multipoles will eventually replace the CSF as the standard method for quantifying the performance of the visual system.

6.1. Uniqueness. The CSF is a curve, with no unique spatial frequencies at which to test. The spatial frequencies that are relevant for a low vision patient, an infant, or the periphery would be lower than the frequencies appropriate for a normal observer. In contrast to the sinusoids, the same five multipole stimuli would be used by all observers since they are unique and discrete. Consider the line multipole. A single high contrast line can be used to test everybody simply by varying the distance. A person with reduced vision would have to get much closer to see the line. Distance can be used to vary multipole strength for all multipoles except the edge. Since the basic stimuli are the same for everyone it is much easier to establish norms and databases. In this regard the multipoles are similar to Snellen acuity.

6.2. Joint-norms. An advantage of a small number of standardized test patterns is that norms for joint probabilities can be developed. Knowing that a patient has low edge sensitivity may be interesting, but knowing that the patient has both low edge sensitivity and normal line sensitivity is much more informative. The development of norms for joint probabilities is made easier when the test functions are small in number and unique.

6.3. Local information. There are many occasions both in the clinic and in basic research when one would like to measure the sensitivity of a tiny spatial region. The multipoles, being localized to a region smaller than the eye's line-spread function, make them ideal for this purpose.

6.4. Phase. Multipole stimuli can be used to obtain a discrimination threshold (the contrast at which a "white" multipole is just distinguishable from a "black" multipole) in addition to the detection threshold (the contrast at which a multipole is just distinguishable from a blank). This phase acuity can not be assessed using simple sinusoids. Paul, Levi and Klein have found that amblyopes show a greater loss in the phase discrimination task than in the detection task.
6.5. Hyperacuity and resolution. The dipole threshold is closely related to the hyperacuity threshold for lines and the resolution threshold of edges, both theoretically and experimentally. Similarly, the quadrupole threshold is closely related to the hyperacuity threshold for dipoles and the resolution threshold of lines. These results were discussed above. It might well be that knowledge of the multipole thresholds will make hyperacuity and resolution thresholds redundant. Further work is needed to determine how closely multipole detection, hyperacuity and resolution covary in a variety of visual systems including those with reduced vision.

6.6. Stimulus generation. It is difficult to generate small patches of high spatial frequency sinusoids without low spatial frequency artifacts at the edges. Pure multipoles, on the other hand, are relatively easy to generate because they are local and can be generated with only two luminance levels (except for the ramp multipole). Details on stimulus generation were discussed earlier.

6.7. Establishing standards. Snellen acuity is a popular clinical test because the 20/20 standard is robust over a wide range of test conditions. It has been difficult to set standards for the CSF because different test conditions produce different results. One factor in CSF variability is that the CSF depends on the number of cycles displayed. Since different test instruments have different field sizes the number of cycles varies for different tests. An advantage of using multipoles is that the number of cycles is fixed. Thus multipole sensitivity should be easier to standardize.

7. CONCLUSION

This article has presented a formalism together with examples of how multipole stimuli can be used in vision studies. Any stimulus that is smaller than the point spread function can be written as a sum of point multipoles. Multipoles are not really new to vision research. Ramps, edges, lines, dipoles and quadrupoles have been used in many previous studies. What is new, however, are many intriguing relationships among the multipoles and between them and other stimuli including sinusoids. The ratio of thresholds of successive multipoles is equal to the Ricci's integration area (Eq. 9). The ratio of sinusoid threshold to multipole threshold gives the bandwidth of the underlying mechanism (Eq. 16). The ratio of a multipole threshold to the strength of a pedestal gives either a hyperacuity threshold (Eq. 25) or a resolution threshold (Eq. 30) depending on whether the difference in orders of the test and pedestal is one or two. One of our cute results is the finding that when expressed in multipole units hyperacuity is poorer than resolution, contrary to popular opinion. The last section of this paper summarized some practical advantages of using a multipole battery for assessing visual performance. As we have gained experience in working with multipoles our appreciation of them has increased and we hope others will come to appreciate them also.

8. ACKNOWLEDGEMENTS

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9. MULTIPOLe MANuSCRIPTS IN PREPAREDnT

2. S.A. Klein and J. Eastlake (1990), "Spatial summation of same and opposite polarity multipoles."

UPDATE on references
1. I hope to submit this during the coming year.
2. I hope to submit this during the next few years.