Minimizing and maximizing the joint space–spatial frequency uncertainty of Gabor-like functions: comment

Stanley A. Klein and Brent Beutter

School of Optometry, University of California, Berkeley, Berkeley, California 94720

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We dispute the claim that Hermite functions (similar to derivatives of Gaussians) minimize a joint uncertainty relation in space and spatial frequency. These functions are found to maximize rather than minimize the uncertainty of the class of functions consisting of an mth-order polynomial times a Gaussian.

Daugman[1] argued that one of the beauties of Gabor functions is that they are compact in both space and spatial frequency since they minimize the Heisenberg uncertainty in space and spatial frequency. The goal is to minimize the product of the variance of the receptive field in space times its variance in spatial frequency. Stork and Wilson[2] subsequently showed (correctly) that Daugman's argument was flawed since it applied only to the complex Gabor functions, whereas in vision research real-valued functions are used. Gabor[3] claimed to show that Hermite functions were the appropriate functions for minimizing the uncertainty relations for real functions. We disagree with this finding and will argue that the Hermite functions produce a *maximum* of the Heisenberg uncertainty conditions for a constrained set of functions consisting of a fixed-order polynomial times a Gaussian. Furthermore, we provide evidence that the Gabor functions do a good job of minimizing an alternative uncertainty relation, contrary to what is claimed by Stork and Wilson.[2] This paper examines a wide class of functions that are localized in both space and spatial frequency. Since all these functions crudely resemble Gabor functions, we call them Gabor-like functions.

The enterprise of Gabor[3] and Stork and Wilson[2] is to find a function $r(x)$ that minimizes the joint space–spatial frequency Heisenberg quantity:

$$U^2 = \langle x^2\rangle\langle f^2\rangle \tag{1}$$

$$= \int_{-\infty}^{\infty} x^2 r^2(x)\mathrm{d}x \int_{-\infty}^{\infty}\left(\frac{\mathrm{d}r}{\mathrm{d}x}\right)^2\mathrm{d}x \Big/ \left[\int_{-\infty}^{\infty} r^2(x)\mathrm{d}x\right]^2. \tag{2}$$

The second integral in Eq. (2) is obtained by using Parseval's theorem to replace the spatial frequency variance $\int_{-\infty}^{\infty} f^2 R(f)^2 \mathrm{d}f$ with $\int_{-\infty}^{\infty} (\mathrm{d}r/\mathrm{d}x)^2\mathrm{d}x$, where $R(f)$ is the Fourier transform of $f(x)$. Gabor[3] attempted to minimize Eq. (2) by using the calculus of variations. He claimed that the Hermite functions produced the minimum. The Hermite functions are given by

$$H_m(x) = \sigma^{m-0.5}(\pi^{0.5}2^m m!)^{-0.5}\exp(x^2/2\sigma^2)D^m\exp(-x^2/\sigma^2), \tag{3}$$

where D^m represents the mth derivative and σ is the standard deviation of the Gaussian. These functions have been normalized:

$$\int_{-\infty}^{\infty} H_m^2(x)\mathrm{d}x = 1. \tag{4}$$

The orthonormality of the Hermite functions allows any function $r(x)$ that is well behaved at $x = 0$ to be expanded in a series of Hermite functions:

$$r(x) = \sum_{m=0}^{\infty} a_m H_m(x). \tag{5}$$

By using Hermite polynomial recursion relations, the joint uncertainty, Eq. (2), can be written in terms of the expansion coefficients, a_m,

$$U^2 = \left\{\left[\sum_{m=0}^{\infty} a_m^2(m+0.5)\right]^2 - \left[\sum_{m=1}^{\infty} a_{m-1}a_{m+1} \times (m^2+m)^{0.5}\right]^2\right\} \Big/ \left(\sum_{m=0}^{\infty} a_m^2\right)^2. \tag{6}$$

To get a clearer understanding of the joint uncertainty it is useful to restrict Eq. (6) to the case in which only two coefficients are nonvanishing. We can then make the general transformation:

$$a_m = A\cos(\theta),$$
$$a_n = A\sin(\theta). \tag{7}$$

The joint uncertainty becomes

$$U^2 = [\cos^2(\theta)(m+0.5) + \sin^2(\theta)(n+0.5)]^2 - \cos^2(\theta)\sin^2(\theta)[(m+1)n\delta(m+2-n) + (n+1)m\delta(n+2-m)], \tag{8}$$

where the last term with the Kronecker delta contributes only if $|n - m| = 2$.

The joint uncertainty, U, of Eq. (8) is plotted in Fig. 1 for $m = 2$ and n ranging from 0 to 5. The horizontal axis is θ, which goes from $-\pi/2$ to $+\pi/2$. If $|n - m| \neq 2$, the curves in Fig. 1 have the simple form

$$U = m + 0.5 + \sin^2(\theta)(n - m). \tag{9}$$

One might be tempted to say that at $\theta = 0$, U is a maximum for $n < m$ and a minimum for $n > m$. This holds

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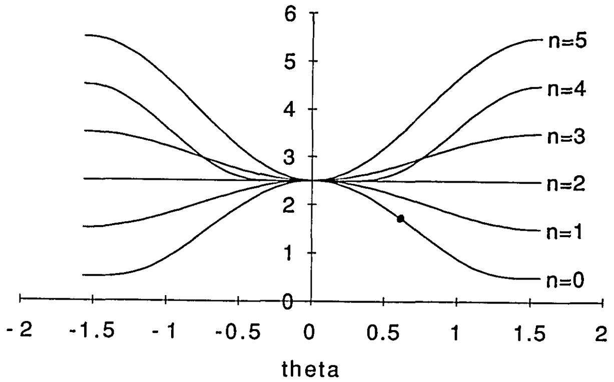

Fig. 1. Plots of the uncertainty U, given in Eq. (2), for the sum of two Hermite functions: $r(x) = \cos(\theta)H_2(x) + \sin(\theta)H_n(x)$. For small values of θ, $r(x)$ is dominated by the second-order Hermite function, and for $\theta \approx \pi/2$, it is dominated by the nth Hermite function. For a single Hermite function, $H_n(x)$, $U = n + 0.5$. For $n < 2$ and also for $n = 4$, U reaches a maximum value of 2.5 at $\theta = 0$. For other values of n, U has a minimum at $\theta = 0$. For the class of functions that are polynomials of a fixed order times a Gaussian, this figure implies that the Hermite functions maximize the uncertainty. The dot on the $n = 0$ function is the combination of H_2 and H_0 corresponding to the second derivative of a Gaussian that had been conjectured to be a minimum of the joint uncertainty.

except for $n = m + 2$. In Fig. 1, the case for $m = 2$, $n = 4$ is seen to have a shape different from those of the others. This is because the negative contribution of the extra term in Eq. (8) is strong enough that $\theta = 0$ becomes a maximum rather than a minimum. For the case of $n = m + 2$ the minimum is at

$$\sin^2(\theta) = (m - 1)m/2(6 + 3m + m^2). \quad (10)$$

The value at the minimum is

$$U^2 = 3(1 + m)(2 + m)(1 + 3m + m^2)/4(6 + 3m + m^2). \quad (11)$$

Figure 1 and Eqs. (6) and (8) show that $\theta = 0$ is a saddle point. That means that if a stimulus is a Hermite function to first order,

$$r(x) \approx H_m(x) + \epsilon f(x), \quad (12)$$

then the uncertainty is of second order in ϵ:

$$U = m + 0.5 + \epsilon^2 K, \quad (13)$$

where K is a finite quantity that can be either positive or negative. If the class of functions is limited to a Gaussian times an mth-order polynomial, then assuming a_n/a_m is of order ϵ, Eq. (6) becomes

$$U = m + 0.5 - \sum_{n=0}^{m-1} (a_n/a_m)^2(m - n) - 0.5(a_{m-2}/a_m)^2 \times m(m - 1)/(m + 0.5) + O(\epsilon^3). \quad (14)$$

Equation (14) shows that the mth-order Hermite function is not merely a saddle point. Gabor was correct that the calculus of variations implies that the Hermite function is an extremum (minimum, maximum, or saddle point). The funny and surprising result of our analysis is that instead of finding a minimum as he had assumed, Eq. (14) shows that the mth Hermite function is the function that *maximizes* the uncertainty! This is because any nonzero value of a_n for $n < m$ will decrease the uncertainty, U.

The precise claim of Stork and Wilson[2] is slightly different. They made an arithmetic error of a factor of 2 in their derivation (as pointed out by Yang[4]) and concluded that it is the family of derivatives of Gaussians that minimize the uncertainty. The derivatives of Gaussians are given by

$$DG_m(x) = D^m \exp(-x^2/2\sigma^2). \quad (15)$$

Based on the normalization of Eq. (3), the second Gaussian derivative can be written in the form of Eqs. (5) and (7) as a sum of the second and zeroth Hermite functions:

$$DG_2(x) = A[(2/3)^{0.5}H_2(x) + (1/3)^{0.5}H_0(x)], \quad (16)$$

where $\cos^2(\theta) = 2/3$.

The uncertainty from Eq. (8) is

$$U^2 = (2/3 \times 2.5 + 1/3 \times 0.5)^2 - 2/9 \times 2$$
$$= 35/12, \quad (17)$$

or

$$U = 1.7078, \quad (18)$$

which is less than the uncertainty of 2.5 for the second-order Hermite function. This point is indicated by the dot in Fig. 1. However, the fact alone that the Gaussian derivatives have a smaller uncertainty than the Hermite functions does not make them special. As seen in Fig. 1, they do not minimize the uncertainty, contrary to the claim of Stork and Wilson.[2] Only the Hermite functions are singled out in that they are saddle points, as shown in Fig. 1.

There is a second error in the paper of Stork and Wilson[2] that is quite interesting. They examine different metrics for the uncertainty relationship. One of their most interesting metrics can be written in the frequency domain as

$$U_{f_o}^2 = \int_0^\infty (f - f_o)^2|R(f)|^2 df \times \int_{-\infty}^\infty [xr(x)]^2 dx \Big/ \left[\int_0^\infty |R(f)|^2 df\right] \times \left[\int_{-\infty}^\infty r^2(x) dx\right]$$
$$= \int_0^\infty (f - f_o)^2|R(f)|^2 df \int_{-\infty}^\infty \left|\frac{dR}{df}\right|^2 df \Big/ \left[\int_0^\infty |R(f)|^2 df\right] \times \left[\int_{-\infty}^\infty |R(f)|^2 df\right], \quad (19)$$

where $R(f)$ is the Fourier transform of $r(x)$. We will call Eq. (19) the displaced uncertainty metric. Equation (19) is the same as Stork and Wilson's Eq. (A12).[2] This metric calculates the frequency variance around a specific spatial frequency rather than around zero. Before reading Stork and Wilson's paper, we assumed that the Gabor function minimized this metric. Stork and Wilson,[2] however, argue that instead of the Gabor function, the function that minimized Eq. (19) has the following tuning in frequency:

$$G_{S\text{-}W}(f) = \exp[-(|f| - f_c)^2\sigma^2/2], \quad (20)$$

where the carrier frequency, f_c, is equal to the displaced frequency in Eq. (19). We worried that these functions have a sharp cusp at $f = 0$ that might cause broadening in space. We therefore numerically calculated the uncer-

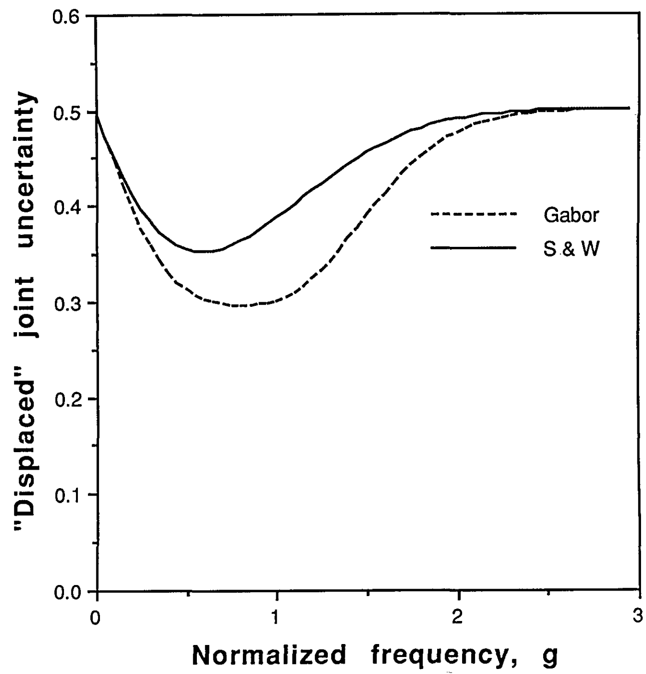

Fig. 2. Pair of plots of the displaced uncertainty U_g as a function of the normalized center spatial frequency, $g = f_o\sigma$, on the abscissa. The dashed curve is for the symmetric Gabor function $G_g(f) = \exp[-(f - g)^2/2] + \exp[-(f + g)^2/2]$. The solid curve is for the function suggested by Stork and Wilson: $G_{S\text{-}W} = \exp[-(|f| - g)^2/2]$. It is seen that, contrary to Stork and Wilson's claim, the uncertainty for $G_{S\text{-}W}$ is larger than that for the Gabor function. The joint uncertainty is always less than $U = 0.5$, the value that pertains to a Gaussian and to a complex Gabor function. The value of 0.5 is the minimum uncertainty of the undisplaced uncertainty, U_0.

tainty, Eq. (19), of both the function of Stork and Wilson and a symmetric Gabor function. In space, the symmetric Gabor function is given by

$$g(x) = \cos(f_c x)\exp(-x^2/2\sigma^2). \tag{21}$$

In spatial frequency it is

$$G(f) = \exp[-(f - f_c)^2\sigma^2/2] + \exp[-(f + f_c)^2\sigma^2/2]. \tag{22}$$

Figure 2 shows plots of the uncertainty, U_g, for the tuning curves given by Eqs. (20) and (22), with normalized frequency units, given by $g = f_o\sigma$, being used for the abscissa. We have taken the carrier frequency, f_c, to equal the displaced frequency, f_o, as was done by Stork and Wilson. The plots show that, contrary to the suggestion of Stork and Wilson, the original Gabor function of Eq. (22) does a better job of minimizing the uncertainty relationship than does their suggestion in Eq. (20). The plots do not give the global minima. We know that forcing the Gabor carrier frequency, f_c, to equal the displaced frequency, f_o, does not produce the minimum uncertainty. In fact, the minimum is achieved when the displaced frequency is given by the mean frequency of the mechanism, as given by

$$f_o = \int_0^\infty f|R(f)|^2 \mathrm{d}f \bigg/ \int_0^\infty |R(f)|^2 \mathrm{d}f. \tag{23}$$

This mean frequency is always above the carrier frequency. Further research is needed to determine what functions produce a global minimum of Eq. (19).

For a zero peak frequency, $g = f_o = 0$, the displaced uncertainty is $U = 0.5$. This was expected since the uncertainty relation of Eq. (19) is the same as the original uncertainty given by Eq. (2) and since the Gabor function for $f_o = 0$ is a Gaussian. The fact that the uncertainty is 0.5 for large values of f_o is a consequence of two ingredients. (1) Since the first integral in Eq. (19) goes over only positive frequencies and since f_o is large, the negative frequency contribution becomes negligible, and so the frequency tuning curve is essentially a Gaussian [the second term of Eq. (22) is negligible]. (2) The spatial variance is given by the variance of the envelope of the Gabor function, which is a Gaussian [Eq. (21)]. These two factors imply that the relevant aspects of the function are Gaussian in both space and spatial frequency so that the uncertainty is 0.5, as for a Gaussian. For $g = f_o\sigma \approx 0.8$ the uncertainty has a minimum value of $U \approx 0.296$ for the Gabor function. This seems to violate the minimum value of $U = 0.5$ associated with the Heisenberg uncertainty relation. There is no real violation, however, since the displaced uncertainty relation of Eq. (19) differs from the Heisenberg relation of Eq. (2). The reduced uncertainty results from the oscillating receptive field's causing the spatial variance to be less than the variance of the Gaussian envelope.

The above integrations were done with a linear frequency axis. Klein and Levi[5] showed that for log axes (as might be appropriate for vision modeling) the Gabor functions had an infinite variance in spatial frequency. Therefore many functions with sharp low-frequency attenuation would do better than the Gabor functions when the uncertainty is calculated by using logarithmic axes.

Minimizing the time–frequency uncertainty has been of interest to researchers in signal analysis from the time of Gabor's paper[3] to the present.[6] One might wonder why the enterprise of minimizing the joint space–spatial frequency uncertainty, U_{f_o}, is of interest to the vision community. The viewprint calculations of Klein and Levi[5] provide some justification. A viewprint is a joint space–spatial frequency plot of the activities of mechanisms that are localized in both space and spatial frequency. If they were not localized, then the mechanisms used both by the visual system and by modelers would be susceptible to masking by stimuli that are well separated in position or in spatial frequency. This localization in space and spatial frequency is precisely what is measured by the displaced uncertainty metric. The Heisenberg metric [Eq. (2)], on the other hand, measures the spatial frequency variance around zero frequency rather than measuring the mechanism bandwidth. Research in image compression provides even stronger reasons for minimizing the joint uncertainty. Localization in space is needed to avoid masking by adjacent features, and localization in spatial frequency is needed to guarantee that the high-pass filters do not have significant low spatial frequencies that would aid their visibility.[7] Most of the information in a scene is stored in the tiny high-spatial-frequency mechanisms, and the main compression savings comes from coarsely quantizing their response (high spatial frequencies have poor visibility). We had hoped that the Gabor function with $g = 0.8$, corresponding to the minimum point of Fig. 2, would be a good candidate for an image compression high-pass filter, but it turns out to have barely any inhibitory side lobes and thus is not a high-pass filter at all.

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