Inverting a Laplacian Topography Map

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Summary: There are some occasions when it is better to use the Laplacian of the scalp voltages (e.g., for topographic displays and dipole source localization) and other occasions when it is better to use the original voltages (e.g., for comparison to non-Laplacian topography maps). This paper presents a simple algorithm for going between these two representations of the data. The inverse Laplacian involves iterative feedback. An Excel spreadsheet implementation of the algorithm is presented.

Key words: Topography; Laplacian; Inverse laplacian; Principal components.

Introduction

Many researchers who measure the topography of evoked potentials choose to work with the Laplacian of the potentials rather than the potentials directly (Srebro 1985; McKay 1984; Gevins 1989). Some arguments in favor of the Laplacian are controversial (see the interchange between Srebro 1992 and van Dijk and Spekreijse 1992 about the Laplacian giving an estimate of current source density) but others are sound (Nunez 1981). For example, by using the Laplacian one avoids the confusions caused by an active reference electrode (Nunez et al. 1991).

An important advantage of the Laplacian is that it is more highly localized than the original potential (see figure 2 of van Dijk and Spekreijse 1992). The tighter localization is useful for topographic maps to better indicate the approximate location of underlying sources. In a recent unpublished study, Carney and I found that the Laplacian is also useful for minimizing errors in the oblique rotation following principal components analysis of multisource responses. After the oblique transformation one ends up with Laplacian maps of the sources. One might want to invert the Laplacian to get a potential map of each component. This paper shows that taking the inverse is surprisingly simple.

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The Laplacian

The first step is to define the Laplacian. Suppose we would like to know the Laplacian at electrode position (x_a, y_a) when that electrode is surrounded by four other electrodes at positions (x_b, y_b), (x_c, y_c), (x_d, y_d), (x_e, y_e). If the electrodes are arranged in an equally spaced checkerboard arrangement then the Laplacian is given by:

$$L(x_a, y_a) = V(x_a, y_a) - (V(x_b, y_b) + V(x_c, y_c) + V(x_d, y_d) + V(x_e, y_e))/4 \quad (1)$$

where V(x, y) is the voltage at position (x, y). If the electrodes are not evenly spaced then the Laplacian can be written as:

$$L(x_a, y_a) = V(x_a, y_a) - \beta V(x_b, y_b) - \gamma V(x_c, y_c) - \delta V(x_d, y_d) - \varepsilon V(x_e, y_e) \quad (2)$$

The coefficients β, γ, δ, ε can be determined by placing several conditions on the Laplacian. First we would like the Laplacian to vanish for a voltage map that is either constant or is increasing linearly. Mathematically the conditions for vanishing can be written as:

$$1 - \beta - \gamma - \delta - \varepsilon = 0$$
guarantees vanishing for V = constant (3a)

$$x_a - \beta x_b - \gamma x_c - \delta x_d - \varepsilon x_e = 0$$
guarantees vanishing for V proportional to x (3b)

$$y_a - \beta y_b - \gamma y_c - \delta y_d - \varepsilon y_e = 0$$
guarantees vanishing for V proportional to y (3c)

Equations 3 a, b, and c are three equations for the four unknowns β, γ, δ and ε. A fourth equation can be obtained by minimizing the variance of the four unknown parameters. Given that the sum of the four parameters is fixed by equation 3a, minimizing the variance is equivalent to minimizing the sum of squares:

$$S = \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 \quad (4)$$

It is easy to check that if the central electrode is at the bisection point of the other electrodes then $\beta = \gamma = \delta = \varepsilon =$ .25 as in equation 1. If a given electrode is surrounded by three rather than by four electrodes then the coefficient ε can be set to zero and the fourth equation is not needed. If an electrode is surrounded by two others along a line then only two parameters (β and γ) and only two equations (equations 3a and b) would be needed. This latter situation would be a common occurrence along an outside edge of the electrode array as will be shown in our simulations.

Inverting the Laplacian

Equations 2 - 4 show that it is straightforward to go from the potentials V(x, y) to the Laplacian L(x, y). The inverse operation might at first sight seem to be quite difficult because it is similar to solving a two-dimensional differential equation (on our discrete lattice it would be a difference equation) with non-uniform sampling points and with special boundary conditions (to be discussed). The solution is easier to achieve than might be expected. It is given by solving equation 2 for $V(x_a, y_a)$:

$$V(x_a, y_a) = L(x_a, y_a) + \beta V(x_b, y_b) + \gamma V(x_c, y_c) + \delta V(x_d, y_d) + \varepsilon V(x_e, y_e) \quad (5)$$

This solution was discussed by Marr (1974) who also pointed out that the Laplacian could be inverted even with arbitrary weightings (and also with less than or more than four surrounding "electrodes").

Equation 5 should look funny to the reader since the unknown voltage V is on both sides of the equation. One might wonder how can one solve for $V(x_a, y_a)$ in terms of the surrounding values of V, which are unknown. The answer is that this is an iterative solution, called a "relaxation method". It is more accurate to write equation 5 as:

$$V_{i+1}(x_a, y_a) = L(x_a, y_a) + \beta V_i(x_b, y_b) + \gamma V_i(x_c, y_c) + \delta V_i(x_d, y_d) + \varepsilon V_i(x_e, y_e) \quad (6)$$

The values V_i on the right side of equation 6 are the voltages on the ith iteration while the value V_{i+1} on the left is the voltage on the i+1st iteration. When the iterations converge, then $V_{i+1}(x,y) = V_i(x,y)$ and equation 5 is obtained, which is the same as equation 2. The solution of equation 6 is unique up to the addition of a linearly changing voltage. This ambiguity is eliminated by having boundary conditions by which the voltage is fixed at a minimum of three independent points, as will be discussed next in connection with an example.

Computer spreadsheets can do the iteration automatically. To illustrate this calculation using the Excel (Microsoft Corp., Redmond, Washington) spreadsheet we provide an example in table 1 of an array of 20 electrodes arranged in 5 rows of 4 columns. Rows 1 - 5 of the spreadsheet are the raw data. The numbers represent voltages at 20 locations on the scalp. Rows 7 - 11 give the formulae for the Laplacians at each electrode.

At the four corners we do not take the Laplacian but rather simply take the raw data. This is done in order to provide boundary conditions for the inverse Laplacian. Thus the entry in cell A7 is the same as the voltage in cell A1. If one chooses the voltage at one of the electrodes to be zero then that electrode is the reference electrode. A minimum of three electrodes must be chosen with specified voltages since equations 3a-c specify that there is a three dimensional space of voltages for which the Laplacian is unchanged.

The entries along the edges are one-dimensional Laplacians. Consider, for example, the entry in cell B7 which can be written in the form of equation 2:

$$B7 = B1 - \beta^*A1 - \gamma^*C1 \quad (7)$$

If we assume that electrode B1 (we are labelling electrodes by their cell numbers) is 60% of the distance from A1 to C1 then equations 3a and 3b lead to $\beta = .4$ and $\gamma = .6$, or

$$B7 = B1 - .4^*A1 - .6^*C1. \quad (8)$$

The six entries in the middle of the array are two-dimensional Laplacians. In this example the coefficients have been written at random, with the only constraint being that equation 3a is satisfied.

Table 1. The values in rows 1 - 5, column A - D are hypothetical voltages at 20 electrodes. The entries in rows 7 - 11 are formulae for Laplacians of the voltage data. The voltages in rows 1 - 5 were chosen to produce a vanishing Laplacian everywhere except at cell B8 at which point the Laplacian is unity. The four corner entries are "anchor" points, not Laplacians, so that the entry in cell D7 is the voltage in cell D1. Three of these anchor points are chosen to have zero voltage and the fourth is given a unity voltage. The formulae for Laplacians are based on the spacing of the electrodes. If the four surrounding electrodes are an equal distance away from the central electrode, then the weighting factors would be $\beta = \gamma = \delta = \varepsilon = 0.25$. For unequally spaced electrodes the weightings are no longer equal. They must still summate to unity in order to guarantee that the Laplacian will not respond to uniform fields. The entries in rows 13 - 17 are formulae for the inverse Laplacian. Again the four corner entries are the anchor point voltages as before. The remaining values have input from the corresponding Laplacian values from rows 7 - 11, summed with neighboring values from rows 13 - 17. The weightings are the same as the Laplacian weightings from rows 7 - 11, except the signs are positive. This positive feedback network requires a number of iterations for convergence. When convergence is achieved the values in rows 13 -17 will equal the values in rows 1 - 5.

	A	B	C	D
1	0	0	0	0
2	0	1.40	.86	.16
3	0	.76	.65	.52
4	0	.37	.44	.76
5	0	.33	.67	1.0

6 LAPLACIAN

7	=A1	=B1 -.4*A1 -.6*C1	=C1 -.8*B1 -.2*D1	=D1
8	=A2 -.2*A1 -.8*A3	=B2 -.4*A2 -.2*C2 -.1*B1 -.3*B3	=C2 -.5*B2 -.2*D2 -.1*C1 -.2*C3	=D2 -.7*D1 -.3*D3
9	=A3 -.3*A2 -.7*A4	=B3 -.1*A3 -.4*C3 -.3*B2 -.2*B4	=C3 -.2*B3 -.3*D3 -.3*C2 -.2*C4	=D3 -.4*D2 -.6*D4
10	=A4 -.5*A3 -.5*A5	=B4 -.2*A4 -.2*C4 -.2*B3 -.4*B5	=C4 -.4*B4 -.1*D4 -.2*C3 -.3*C5	=D4 -.5*D3 -.5*D5
11	=A5	=B5 -.5*A5 -.5*C5	=C5 -.5*B5 -.5*D5	=D5

12 INVERT LAPLACIAN

13	=A7	=B7 +.4*A13 +.6*C13	=C7 +.8*B13 +.2*D13	=D7
14	=A8 +.2*A13 +.8*A15	=B8 +.4*A14 +.2*C14 +.1*B13 +.3*B15	=C8 +.5*B14 +.2*D14 +.1*C13 +.2*C15	=D8 +.7*D13 +.3*D15
15	=A9 +.3*A14 +.7*A16	=B9 +.1*A15 +.4*C15 +.3*B14 +.2*B16	=C9 +.2*B15 +.3*D15 +.3*C14 +.2*C16	=D9 +.4*D14 +.6*D16
16	=A10 +.5*A15 +.5*A17	=B10 +.2*A16 +.2*C16 +.2*B15 +.4*B17	=C10 +.4*B16 +.1*D16 +.2*C15 +.3*C17	=D10 +.5*D15 +.5*D17
17	=A11	=B11 +.5*A17 +.5*C17	=C11 +.5.*B17 +.5*D17	=D11

The original voltages in rows 1 - 5 were chosen so that the Laplacian values are simple. All the Laplacian values in rows 7 - 11 are zero except for cell B8 which is unity! In addition, the four corner boundary conditions are taken to be zero except for the lower right corner (cell D5) which is unity. The values in row 5 are especially easy to understand because the Laplacian weightings in cells B11 and C11 are all .5. This choice of weightings causes the values in row 5 to be linearly increasing as a function of position. Notice that the largest voltage occurs in cell B2, corresponding to the position where the Laplacian was non-zero.

The formulae for the inverse Laplacian are given in rows 13 - 17 of the table. These entries are similar to the Laplacian in rows 7 - 11 except that the minus signs become plus signs. For example, the corner elements are fixed to be equal to the original voltages. These four values provide the anchor points that are commonly called the "boundary conditions". The inverse Laplacian corresponding to the electrode at position B1 is:

$$B13 = B7 + .4*A13 + .6*C13. \quad (9)$$

The inverse Laplacian for the neighboring position C1 is:

$$C13 = C7 + .8*B13 + .2*D13 \quad (10)$$

As discussed earlier, this might seem like an illegal set of equations since the equation for B13 involves the value at C13 and the equation for C13 involves the value for B13. However, to my delight, Excel had no problem coping with this self-referential situation. Excel has an option for iterating either for a fixed number of iterations or until a convergence criterion is met. When Excel stopped iterating, it was pleasing to discover that the values in rows 13 - 17 were identical to the values in rows 1 - 5 (in its normal mode of operation Excel shows the values in each cell rather than the equations). This verifies that we have indeed found the inverse Laplacian.

As was discussed in the Introduction there are occasions when it is advantageous to do early stages of data analysis using the Laplacian and later stages using voltages at each electrode. For this type of processing, a method such as the one described in this paper, is needed for inverting the Laplacian. This method only works if all the coefficients for taking the original Laplacian are known. In order to be able to have both the Laplacian map and the direct voltage map, it makes sense to digitize the voltage at each electrode and then calculate the Laplacian by software as in equation 2. The alternative method is to directly measure the Laplacian by hardware (where three or more electrodes are differentially amplified in center-surround fashion to produce a single response). The software method has the advantage that all the coefficients and boundary values are known, enabling one to go back and forth between the direct voltages and the Laplacian voltages.

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