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Axial and Instantaneous Power Conversion in Corneal Topography

Stanley A. Klein and Robert B. Mandell

Purpose. To determine if a direct relationship exists between the two common measures of corneal topography—axial and instantaneous powers—that are based on corneal slope or curvature, respectively.

Method. The theoretical relationship between axial and instantaneous powers was derived for unrestricted shapes using equations of basic calculus.

Results. It was found that axial power at any point y on the cornea is exactly equal to the average of the instantaneous powers from the axis to point y.

Conclusions. A simple relationship exists between axial and instantaneous powers that is valid for the intersection of any surface by a meridional plane. This provides a practical means for converting between axial and instantaneous powers for clinical applications of corneal topography. Invest Ophthalmol Vis Sci. 1995;36:2155-2159.

In clinical measures of corneal topography, the shape of the corneal surface is usually interpreted from an array of discrete measures using the paraxial power formula for a single refracting surface, $P = (n - 1)/r$ where n is the index of refraction and r is the radius of curvature. Confusion arises because away from the optical axis, the radius may be measured as either of two values, the instantaneous radius of curvature, r_i (see the mathematical expression in equation 10), or the axial distance, d_a, which is the perpendicular distance from the corneal surface to the optical axis.[1,2] This leads to two expressions of corneal power, namely instantaneous power and axial power.[3] It is recognized that neither axial nor instantaneous power adequately represents corneal refractive power, but each provides shape measures in the form of slope and curvature, respectively. Radius units, rather than power units, might provide a better expression of corneal shape[4]; as will be seen in our derivations, the power units lead to more elegant mathematical expressions.

The most commonly used clinical instrument for measuring corneal topography is the videokeratograph, which traditionally measures axial power, although attempts have been made to measure instantaneous power.[5-8] Other corneal topographers not based on placido disk technology usually provide measurements of the corneal coordinates and generally must convert to a power expression to conform to clinical convention. It is recognized that measurements in terms of either axial or instantaneous power may be preferred for some applications and that an algorithm for converting these powers is of value. Such conversions based on various continuity assumptions have been proposed,[5-9] but this report presents a method for power conversion unlimited by a priori assumptions about corneal shape.

The current analysis considers the curvature and slope of the cornea in a meridional plane containing the reference axis of the measurement system, which is normal to the cornea. Instantaneous power is *defined* as:

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$$P_i = (n - 1)/r_i \tag{1}$$

where r_i is the instantaneous radius of curvature of the surface according to the usual mathematical definition (see equation 10). The interpretation of equation 1 has produced confusion in the field of corneal topography because it is a standard optical formula based on paraxial refraction that is now commonly used to describe shape. We avoid this inconsistency by interpreting equation 1 as a *definition* of instantaneous power that does not have the paraxial limitation. In this report, we use power in terms of dioptric units to describe corneal shape for two reasons: It is already familiar to many researchers, and the equations we derive are simplest when expressed in power units.

Axial power is *defined* by:

$$P_a = (n - 1)/d_a \tag{2}$$

where d_a is the axial distance, the perpendicular distance from the cornea to the optic axis. Because all calculations in this report are for quantities defined in the meridional plane, the normal to the cornea will always intersect the optic axis. The axial distance is given by:

$$d_a = y/\sin(\theta) \tag{3}$$

where y is the height of the corneal point (see Fig. 1), and θ is the angle between the surface normal and the optic axis. Tan(θ) is the slope of the cornea. Axial power may be termed *slope-based power* because the power at point y is determined by the slope at that point. The Appendix provides a method by which the axial power within the meridional plane can be calculated for any point of an arbitrary corneal surface and an arbitrarily positioned axis.

METHODS. Figure 1 shows a cross-sectional view of the cornea with the quantities r_i, d_a, y, and θ (described earlier) and h, the vertical distance of the center of curvature from the optic axis. By eliminating the axial distance from equations 2 and 3, a direct connection is found between the axial power and the slope of the surface:

$$(n - 1)\sin(\theta) = yP_a. \tag{4}$$

The quantity on the left also can be related to corneal curvature by considering the larger triangle in Figure 1:

$$\sin(\theta) = (y - h)/r_i. \tag{5}$$

Taking the derivative of both sides of equation 5

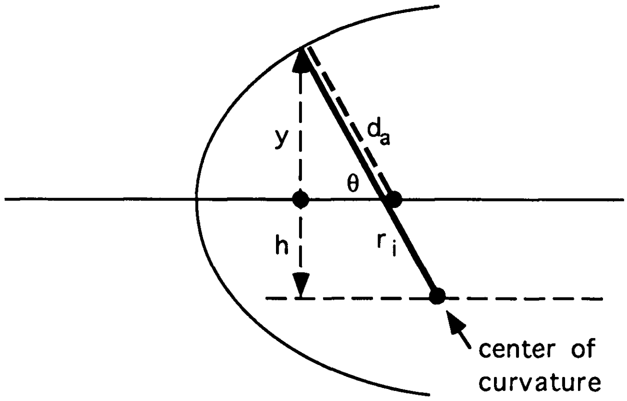

FIGURE 1. A cross-section of the corneal shape together with parameters used to specify corneal power at vertical height, y. This cross-section is called the meridional plane. All our calculations are performed for the corneal profile in this plane. The distance from the cornea to the optic axis along the normal is d_a. The instantaneous radius of curvature is r_i. It is measured along the normal to the cornea. The center of curvature is instantaneously a distance h below the optic axis. The same angle θ is the angle of incidence of an incoming ray, parallel to the axis, that strikes the cornea at a distance y from the optic axis.

with respect to y leads to the following simple result because instantaneously h and r_i are fixed:

$$d\sin(\theta)/dy = 1/r_i. \tag{6}$$

The concept of r_i and h being fixed may be unfamiliar to many readers, so two alternative derivations of equation 6 are presented, the first more formal and the second more intuitive. The formal derivation begins by expressing $\sin(\theta)$ in terms of $\tan(\theta) = dz/dy$ as follows:

$$\sin(\theta) = (1 + \tan^{-2}(\theta))^{-1/2} \tag{7}$$

$$\sin(\theta) = (1 + (dz/dy)^{-2})^{-1/2}. \tag{8}$$

Using the chain rule of calculus, the left side of equation 6 becomes:

$$d\sin(\theta)/dy$$
$$= (1 + (dz/dy)^{-2})^{-3/2}(dz/dy)^{-3}d^2z/dy^2 \tag{9}$$

$$d\sin(\theta) = d^2z/dy^2(1 + (dz/dy)^2)^{-3/2} = 1/r_i. \tag{10}$$

Equation 10 is familiar as the definition of curvature of a general curve. Thus, we have an alternative derivation of equation 6.

The final derivation of equation 6 is based on Figure 2. We start by assuming a small section of the cornea is well approximated by a segment of a circle of radius r_i. Lines are drawn in the normal directions

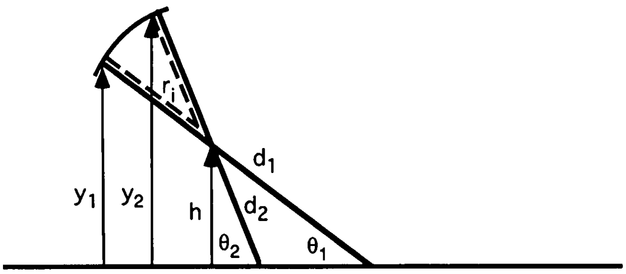

FIGURE 2. A small circular segment whose radius of curvature, r_i, is used to derive the connection between instantaneous and axial powers. The endpoints of the segment are labeled 1 and 2. The distance of the endpoints from the reference axis are y_1 and y_2. The axial distances (in the normal direction) of the endpoints from the reference axis are d_1 and d_2. The angles between the normals at the endpoints and the reference axis are θ_1 and θ_2.

from the ends of the segment to the reference axis. The lengths of these two normals are d_1 and d_2. The distances of the segment ends from the reference axis are y_1 and y_2. The angles between the two normals and the reference axis are θ_1 and θ_2. Simple trigonometry can be used to show that the difference between y_2 and y_1 can be written as:

$$\Delta y = y_2 - y_1 = r_i \sin(\theta_2) - r_i \sin(\theta_1). \qquad (11)$$

or

$$\Delta \sin(\theta)/\Delta y = 1/r_i \qquad (12)$$

where $\Delta \sin(\theta) = \sin(\theta_2) - \sin(\theta_1)$. In the limit, as Δy becomes very small, equation 12 becomes equation 6.

Multiplying both sides of equation 6 by (n-1) to convert from inverse meters to diopters gives the connection between θ and P_i:

$$(n - 1)d \sin(\theta)/dy = P_i. \qquad (13)$$

RESULTS. Combining equations 4 and 13 gives the main result of this report:

$$P_i(y) = dy(P_a(y))/dy \qquad (14)$$

$$P_i(y) = P_a(y) + ydP_a(y)/dy. \qquad (15)$$

In equations 14 and 15, the dependence of power on y has been made explicit with the notation $P(y)$.

The inverse relationship giving P_a in terms of P_i is:

$$P_a(y) = \frac{1}{y}\int_0^y P_i(y')\,dy'. \qquad (16)$$

Equation 16 shows that the axial power at position y is equal to the average of the instantaneous powers over the entire interval from 0 to y. This result is not surprising because instantaneous and axial powers are closely related to the second and first derivatives of corneal shape. What might be unanticipated is that the connection is direct, as given by the exact solutions in equations 14 to 16.

To make the connection between axial and instantaneous power concrete, it is useful to consider an example in which a section of the cornea is a circular segment so that the instantaneous radius of curvature is fixed. From Figure 1 (or equations 3 and 5), we find the general relationship:

$$(y - h)/r_i = y/d_a \qquad (17)$$

where h is the distance from the axis to the center of curvature. Using equations 1 and 2 to express r_i and d_a in power units gives:

$$P_a(y) = P_i(y)(1 - h/y). \qquad (18)$$

When the cornea is a circular segment, P_i and h are independent of y so that equation 18 can be written as:

$$P_a(y) = P_i - P_i h/y. \qquad (19)$$

For negative values of h (center of curvature below the reference axis), the axial power is greater than the instantaneous power. When h is positive (the center of curvature is above the axis), there is a point, $y = h$, at which the axial power vanishes because the corneal normal is parallel to the axis. Equation 19 provides a simple connection between axial and instantaneous powers for a circular section of cornea if the displacement of the axis from the center of curvature of the circular patch is known. Klein and Mandell[10] make use of equation 19 for calculating the axial power of a model of keratoconus and a model of photorefractive keratectomy (excimer laser corneal flattening).

DISCUSSION. Equation 16 provides a simple relationship between axial power and instantaneous power: The former is the average of the latter over the interval from the optic axis to the corneal point of interest. This connection is valid for corneas of arbitrary shape as long as the axial and instantaneous powers are defined in a meridional (also called tangential) plane. The only restriction necessary for a meaningful definition of axial power is that the axis be normal to the corneal surface. Nowhere have we made additional assumptions about the surface shape.

For some corneal topographers, corneal shape is specified in Cartesian coordinates, with z expressed as a function of x and y. A conversion to axial radius can be calculated at an arbitrary point and for an arbitrary axis using a method described in the Appendix.

We have favored the use of the terms axial and instantaneous powers instead of tangential and sagittal powers, used by some others, because tangential and sagittal have specific meanings in optics relating to peripheral ray bundles at angles to the optic axis either in the meridional plane or 90° to it, respectively. In corneal topography, there is an axial and an instantaneous power in each of the tangential and sagittal planes. This forms the basis for a future treatment of nonradially symmetric surfaces, for which a multiplicity of definitions can be found for axial and instantaneous powers.[11] For radially symmetric corneal shapes (surfaces of revolution), the axial power in the tangential plane equals both the axial power and the instantaneous power in the sagittal plane. For corneal shapes not radially symmetric, the tangential plane's axial power no longer equals the sagittal plane's instantaneous power. However, the simple relationship between axial and instantaneous powers derived in this report (equations 14 and 16) is still valid for powers defined in the meridional (tangential) plane.

Equation 15 allows us to derive a simple formula for the amount of "corneal cylinder" power for an axially symmetric cornea. The amount of cylinder is the difference between the tangential and sagittal planes' instantaneous powers, corresponding to the difference between the tangential plane's instantaneous and axial powers. From equation 15, the amount of cylinder is:

$$P_{cyl} = P_i - P_a \qquad (20)$$

$$P_{cyl} = y dP_a(y)/dy. \qquad (21)$$

That is, the rate of change of axial power is directly related to the local amount of cylinder power.

One final way of expressing the connection between axial and instantaneous powers is useful. Equation 16 was obtained by integrating equation 14 from 0 to y. The integration could have been achieved over an interval that did not begin at the axis, in which case one obtains:

$$\Delta(yP_a)/\Delta y = P_{i\,\mathrm{AVE}}. \qquad (22)$$

This equation relates the change in axial power across a segment to $P_{i\,\mathrm{AVE}}$, the average instantaneous power within that segment. This relationship, which also can be derived from equation 12, is especially useful when dealing with circular corneal segments for which P_i is constant.

Key Words

cornea, corneal power, corneal topography, videokeratography

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APPENDIX. Calculation of Axial Power for an Arbitrary Corneal Shape. All our calculations have been based on quantities defined in the meridional plane. However, it is possible to start from a surface defined in three dimensions and to find the needed quantities in the meridional plane that can be used to calculate the axial power.[12] To be concrete, the formalism will be applied to an ellipsoid with semiaxes of 7, 8, and 10 mm, given by:

$$(x/7)^2 + (y/8)^2 + (z/10 - 1)^2 = 1 \qquad (23)$$

or

$$z(x, y) = 10(1 - (1 - (x/7)^2 - (y/8)^2)^{1/2}). \qquad (24)$$

The normal at point x, y has a direction given by:

$$\bar{n} = k(-\partial z/\partial x, -\partial z/\partial y, 1) \quad (25)$$

where $\partial z/\partial x$ is the derivative of equation 24 with y held fixed. For the ellipsoid given by equations 23 or 24, the normal is in the direction of:

$$\bar{n} = [x/49, y/64, (z - 10)/100]. \quad (26)$$

Next, calculate the axial power at corneal position

$$(x_c, y_c) = (2, 2) \quad (27)$$

when the axis strikes the cornea at position

$$(x_a, y_a) = (0.5, 1.0). \quad (28)$$

The vector from the vertex to the corneal point is:

$$\bar{r} = (x_c, y_c, z_c) - (x_a, y_a, z_a)$$
$$(1.5. 1.0, 0.6445). \quad (29)$$

From equation 26, the unit normal at the axis (the direction of the axis) is:

$$\hat{n}_a = (0.1013, 0.1552, -0.9827). \quad (30)$$

The normal, n_c, at the corneal point of interest as given by equation 26 is not directly usable because it does not lie in the meridional plane. The projection of the normal onto the plane can be calculated based on the vector, $\bar{M}$, that is normal to the meridional plane:

$$\overline{M} = \hat{n}_a X \bar{r} \quad (31)$$

where X stands for the vector cross-product. In terms of $\bar{M}$, the projection of n_c onto the meridional plane is:

$$\overline{N} = \overline{M}X(\overline{M}Xn_c). \quad (32)$$

The unit vector of the normal in the meridional plane is:

$$\hat{N} = (0.3623, 0.3292, -0.8720). \quad (33)$$

It is now possible to calculate the axial distance using equations 2 and 3.[11] In terms of the vector quantities defined in this Appendix, equation 3 becomes:

$$\begin{aligned} d_a &= y/\sin(\theta) \\ &= |\hat{n}_a X \bar{r}| / |\hat{n}_a X \hat{N}| \\ &= 5.7516 \text{ mm}. \end{aligned} \quad (34)$$

The numerator of equation 34 is y, the projection of r normal to the axis. The denominator of equation 34 is $\sin(\theta)$, which is the magnitude of the cross-product between the unit vector in the axis direction and the corneal normal in the meridional plane. Equation 32 was needed to obtain the normal in the plane. With this formalism, it is straightforward to calculate the axial power for an arbitrary corneal surface and a meridional plane defined by an arbitrary axis. Given the axial power, equation 14 allows the instantaneous power to be calculated in that plane using numerical differentiation.

The definition of axial distance, d_a, given by equation 34 is the definition based on projecting all quantities onto the meridional plane. When the surface normal does not lie in the meridional plane, alternative definitions of axial distance are possible.[11]