Method for Generating the Anterior Surface of an Aberration-Free Contact Lens for an Arbitrary Posterior Surface

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ABSTRACT

Advances in accuracy of corneal topography have made it possible to design contact lenses whose posterior surface is based on the corneal contour. Our goal is to design a contact lens of focal length, $f$, given an arbitrarily shaped posterior surface without assuming rotational symmetry. The shape of the anterior surface is calculated using the principle of least time, thereby going beyond paraxial assumptions and the thin lens approximation. There is no requirement that the reference ray be normal to either the anterior or posterior surface of the lens.

Key Words: corneal topography, contact lens, anterior surface, ray tracing, Fermat's principle, aberrations

The posterior surface of the optic zone of a contact lens is generally spherical, or aspherical but still close to spherical with axial symmetry, or toric. However, we would like to design a contact lens with an arbitrarily shaped posterior surface.

In this article, we calculate the shape of the anterior surface, such that the contact lens has a focal length, $f$, when measured in the eye. In the thin lens approximation, the thickness, $t$, of the contact lens is:

$$t = t_0 - x^2 n_{eye} / 2 f (n_{eye} - 1)$$

where $r = (x^2 + y^2)^{1/2}$ is the distance from the axis, $n_{eye}$ and $n_{contact}$ are the indices of refraction of the contact lens and the eye, and $t_0$ is the thickness at the axis. Notice that equation 1 does not depend on the shape of the anterior or posterior surface. This independence of shape is the result of three approximations:

1. The lens is thin (relative to the radius of curvature of the contact lens).
2. The lens diameter is small (relative to the radius of curvature of the contact lens).
3. The angles between the incoming ray and the normals to the anterior and posterior surfaces are small. Approximations 2 and 3 are commonly called the paraxial approximation.

In this paper, we will go beyond all three approximations so that the lens can be thick, the optical zone can be of any size, and the chief ray need not be normal to either anterior or posterior surface. The shape of the back surface can have a strong effect on the focusing power of the lens and in our method the back surface is taken into account exactly. In order to simplify the mathematics we assume an object near infinity (parallel incoming rays); we discuss how our approach can be extended to designing a lens for an arbitrary object distance.

METHODS

Our goal is to start with an incoming beam of parallel light that is refracted by the lens so that all the rays come to a focus at a distance $f$ from the back of the lens. This is equivalent to assuming zero aberrations for axial rays.

Fig. 1 shows the lens with the variables that we are using to describe the optical path length. The figure is drawn in the $xy$-plane ($z = 0$) with the $x$ axis coming out of the paper. Six points are shown in Fig. 1:

- points A and B where the reference ray hits the anterior and posterior surface, respectively, of the lens
- points D and E are similar to points A and B except for an arbitrary ray
- point C, the point on the arbitrary ray that is on the same wavefront as point A; that is, point C is the same distance from the object as is point A
- point F where all rays come to a focus.

It is important to clarify how the reference ray and lens parameters are chosen and to clarify which of the parameters must be calculated.
The thickness of the contact lens along the reference ray is a free parameter. It is chosen so that the thinnest point of the contact lens is just thick enough for stability. This point is chosen at the origin of the coordinate system. The reference ray is drawn as a solid line until it emerges from the contact lens because it is in the plane of the paper (the yz-plane). We have chosen the coordinate system so that the portion of the reference ray inside the contact lens lies in the yz-plane. After the ray emerges from the posterior surface, it can be drawn in the yz-plane, but additional complexity would be present (see discussion section). Different objects distances correspond to different contact lens powers.

The direction of the normal, $n_A$, at point A on the anterior surface is a free parameter. One possibility is to choose it to be the same direction as the posterior surface normal at point B, so that the reference ray will be undeviated in direction. If prism ballast were needed (to properly orient the lens) then $n_A$ would be appropriately rotated. Our derivation would not be affected.

The six parameters of the coordinate system (three parameters specifying the orientation of the axes) are defined as follows:

- We choose the origin to be point A. Thus, point A has coordinates: $(x_A, y_A, z_A) = (0, 0, 0)$.
- We choose the direction of the $x$ axis to be along the direction of the incoming rays. Thus, the direction of an incoming ray is $q_x = (0, 0, 1)$.
- We choose the direction of the $y$ axis so that the normal direction of the anterior surface at point A is in the yz-plane. Thus, the normal can then be written as: $n_A = (0, a, b)$, where $a$ is a free parameter.

In this coordinate system, points B, D, E, F, and G have coordinates: $(x_B, y_B, z_B)$, $(x_D, y_D, z_D)$, $(x_E, y_E, z_E)$, and $(x_F, y_F, z_F)$.

Quantities that Must be Calculated

The goal of this article is to calculate the thickness of the contact lens, $t_{ray}$, for an arbitrary ray that leaves the posterior surface at point E. The thickness is specified in the direction of that ray inside the contact lens. The goal is to have all rays leaving the object come to focus at the same focal point.
point. Thus, there would be zero monochromatic aberrations for that particular focal point.

To achieve this goal of calculating the lens thickness at an arbitrary point \((x_B, y_B, z_B)\) on the back surface, we must trace a ray from the focal point and through the lens. The direction of the ray inside the contact lens must be calculated by Snell's law, given knowledge of the direction that the ray leaves the lens (it is directed toward the focal point) and knowledge of the normal direction of the posterior surface because that surface is assumed to be known. The angle of the ray inside the lens with respect to the z axis will be denoted \(\theta\).

The direction of an incoming ray, \(\theta\), can be determined by two methods: (1) from Snell's law (assuming we are given the back surface and given that we have calculated the front surface, which is the goal of this article), and (2) by having the ray go from point D to the object point. Both methods define the same curve.

To calculate the thickness of the contact lens, we will use a ray tracing method based on Fermat's principle. According to this principle, also called the principle of least time, all rays take the same amount of time to reach the focal point if all the rays are in focus. Because the time of travel is equal to the distance divided by the velocity, one can define an optical path length as the geometric distance times the index of refraction (the velocity is inversely related to the index of refraction).

The optical path length of the reference ray, denoted \(\text{Length}(ABF)\), is the sum of the path length from A to B plus that from B to F:

\[
\text{Length}(ABF) = n_{\text{refl}}t_{\text{AB}} + n_{\text{refl}}f.
\]

The thickness of the contact lens along the reference ray (the distance from A to B) is:

\[
t_{\text{AB}} = (y_B^2 + z_B^2)^{1/2},
\]

The distance from point B to the focal point, \(F\), is:

\[
f = (x_B^2 + (y_B - y_F)^2 + (z_B - z_F)^2)^{1/2}.
\]

The index of refraction of the eye, \(n_{\text{eye}}\), is assumed to be uniform throughout the eye and the same as that of the tear film. The posterior surface of the tear film fills in most of the small scale irregularities of the corneal surface. The front surface of the tear film coincides with the posterior surface of the contact lens. Because the refractive index of the tears and eye are assumed to be the same, in our calculations the two are combined and we refer to the combination as the anterior surface of the eye. A standard rigid contact lens having a spherical posterior surface could largely compensate for corneal irregularities. However, such a posterior surface is not always able to achieve a satisfactory fit to the cornea. Instead, the shape of the contact lens should conform somewhat to the contour of the cornea. In the case of keratoconus, for example, a contact lens having a spherical posterior surface could rub and irritate the apex of the cone. Determining a good posterior surface shape is an interesting and challenging problem, however, that is not the purpose of this paper. Our goal is to start with a given posterior surface shape and determine the optimal anterior surface shape.

Fig. 1 shows the reference ray entering along the z axis. The reference ray to the left of point B is drawn as a solid line to indicate that it lies in the plane of the diagram, that is, the yz-plane. The reference ray is drawn as a dashed line upon emerging from the back of the contact lens because it would not lie in the yz-plane should the posterior surface normal not lie in that plane. A second ray is also shown; it is drawn as a dashed line to indicate that it does not necessarily lie in the yz-plane. Note that our derivation does not require co-planar rays. Thus, our result for contact lens thickness is valid for contact lenses of general three-dimensional shape.

The optical path length from point C to point F for an arbitrary ray emerging from the posterior surface of the contact lens at a point \((x_B, y_B)\) consists of three parts (C to D, D to E, E to F):

\[
\text{Length}(CDEF) = n_{\text{air}}z_D + n_{\text{eye}}t_{\text{DE}} + n_{\text{air}}d
\]

where \(n_{\text{air}}\) is the refractive index of air, \(z_D\) is the z-position (usually unknown) where the incident ray strikes the anterior surface of the lens measured relative to the origin (the point where the reference ray strikes the anterior surface), \(t_{\text{DE}}\) is the thickness of the contact lens in the direction of the ray within the lens, and the starting point C has coordinates \((x_D, y_D, 0)\) because the ray comes in parallel to the z axis so that it has the same x and y coordinates as point D. Point C is the starting point of the path length calculation because the ray from the object would have the same incoming phase at that point as the reference ray has at point A (both points A and C are at \(z = 0\)).

The distance, \(d\), measured from the focal point, \(F\), to the point \(E\) on the posterior surface is:

\[
d = [(x_E - x_B)^2 + (y_E - y_B)^2 + (z_E - z_B)^2]^{1/2}
\]

where \(z_E\) can be obtained given \(x_E\) and \(y_E\), the chosen coordinates of the ray, because we are assuming that the posterior surface of the contact lens is known.

**RESULTS**

From Fig. 1, the z distance, \(z_B\), between an arbitrary point on the posterior surface and the \(z = 0\) reference plane defined by point A, is given by:

\[
z_B = z_D + t_{\text{DE}} \cos(\theta)
\]
where \( \theta \) is the angle with respect to the incoming direction of the ray within the lens. This angle is calculated by tracing a ray from the focal point to the posterior surface and then using Snell's law for general surfaces. Rearranging equation 7 yields an expression for \( t_{DP} \) in terms of the known quantities, \( z_{2} \) and \( \theta \), plus the unknown thickness, \( t_{DP} \):

\[
t_{DP} = z_{2} - t_{DP} \cos(\theta).
\]

If the object were not at infinity then the segment \( CD \) would not be parallel to the incoming reference ray and equations 7 and 8 would become more complicated. By choosing the object to be infinitely distant, we have simplified the calculation.

The constraint of equal path length means that equation 5 is equal to equation 2:

\[
\text{Length}(ABF) = \text{Length}(CDEF).
\]

By equating equations 2 and 5 and substituting equation 8 for \( z_{2} \), the equal path length equation becomes:

\[
\frac{n_{air} \cdot t_{AB} + n_{ray} r_{F}}{n_{air} \cdot t_{AB}} = n_{air} \cdot t_{DF} (n_{air} \cdot n_{air} \cdot \cos(\theta)) + n_{air} \cdot t_{DF}.
\]

Finally, equation 10 can easily be solved for the thickness, \( t_{DF} \):

\[
t_{DF} = \frac{n_{air} \cdot t_{AB} + n_{ray} (f - d) - n_{air} \cdot t_{DF}}{n_{air} \cdot n_{air} \cdot \cos(\theta)}.
\]

Equation 11 is a simple formula for the lens thickness in terms of quantities that can be calculated based on the posterior surface of the contact lens.

The three coordinates of point D where the ray hits the anterior surface are obtainable from the standard ray tracing equations because the ray direction inside the lens is given by Snell's law and the distance between surfaces in the ray direction is now known (equation 11). The \( z \) coordinate of point D is especially easy to calculate because it is given by equation 8.

It is interesting to note that the thin lens paraxial approximation (equation 1) is a special case of equation 11. Taking \( \cos(\theta) \approx 1 \) and \( z_{2} \approx t_{AB} \) in equation 11, yields:

\[
t_{DF} \approx t_{AB} + n_{ray} (f - d)/(n_{air} \cdot n_{air} - n_{air}).
\]

If the reference ray is chosen to be the optic axis so that it is undeviated, then \( F \) is on the optic axis and \( d \) can be expanded in a Taylor's series (using \( n_{air} = 1 \)) to give the "agg" formula:

\[
d - f \approx (x_{x}^{2} + y_{x}^{2} + f^{2})/2f - f
\]

Substituting this value into equation 12 yields the thin lens approximation (equation 1). This thin lens approximation (equation 14) is often used for calculating the anterior surface of a contact lens. It provides the proper paraxial optics (near the center of the lens) but does not correct the spherical aberration.

DISCUSSION

We have calculated the thickness of a contact lens going beyond paraxial assumptions and beyond the thin lens approximation. The result is to produce an aberration-free image for incoming rays parallel to the axis. Off-axis aberrations such as coma, radial astigmatism, and distortion are still present. The solution for thickness given by equation 11 is an exact equation with no approximations. It gives the contact lens thickness in terms of known quantities, because the posterior surface of the contact lens is assumed to be known.

Equation 11 is simple for three reasons: First, the thickness is specified in the direction of the refracted ray inside the contact lens. This direction is directly calculated from Snell's law because the point on the posterior surface of the contact lens is known and the direction in which the ray leaves the posterior surface is known because the ray goes to the focal point. If the thickness had been specified in terms of the horizontal distance or normal distance, then the formula for \( t_{DF} \) would be much more complicated. Second, the thickness is specified in terms of the coordinates of the posterior surface of the contact lens (which are the same as the coordinates of the cornea) rather than the coordinates of the anterior surface of the contact lens. Third, the object is taken to be at infinity so that the incoming rays are all parallel to the axis. If a different object position had been chosen, then the quantity \( n_{air} \cdot t_{0} \) in equation 8 would be replaced with a Pythagorean sum involving several known quantities and \( n_{air} \cdot t_{0} \) in this case, equation 10 would become a quadratic equation in \( t_{DF} \) rather than the present linear equation.

It is important to address the issue of feasibility of fabrication of such contact lenses. This can be realized using computer numerical control technology, which is capable of producing complicated irregular surfaces. As is discussed by Barsky, computer-controlled machining can manufacture a general complex shape to the desired accuracy according to a mathematical model.

Another question that arises is how sensitive are the aberrations to contact lens decentration (e.g., due to eye blinks) or rotation (e.g., due to ineffective balance). The answer depends on the shapes of the lens and corneal surfaces. If the surfaces are close to being spherical then the effect of decentration and rotation would be small. Calculation of these effects is beyond the scope of this paper.

The calculations presented here are designed to
remove axial aberrations for an object at infinity. If the object is placed at a location that differs from the design location, then a small amount of spherical aberration will be introduced. The methods for calculating the exact, diffraction-limited point spread function and modulation transfer function in the presence of aberrations were presented by Klein and Ho.8

In equation 11 we combined the vitreous humor, crystalline lens, aqueous humor, corneal thickness, and the tear film into a single schematic eye. Our approach would allow these more detailed elements to be added by modifying the optical path lengths r_{opt} and n_{cor} with the correct optical path lengths from the focal point to the corneal points and by changing θ to the correct angle. This more accurate method could be easily implemented on a computer with ray tracing using Snell’s law if the optical components are known. The result would be a general and accurate technique for calculating the thickness of a contact lens that is free of axial monochromatic aberrations. Although precise locations of the corneal posterior surface and properties of the crystalline lens (gradient index) are not known, they can be fairly accurately approximated based on Purkinje images and ultrasound or laser interference measurements. Even approximate estimates of these quantities would improve the quality of the anterior surface calculation. The goal of this paper is to demonstrate how a closed form solution for the contact lens thickness can be calculated. Our calculation method can handle inclusion of multiple ocular surfaces.

One might ask whether it is worthwhile to compensate for the axial monochromatic aberrations. For people with normal vision, compensating the aberrations would not produce substantially better vision. However, individuals with severe aberrations could receive significant benefit from the specially designed contact lens front surface discussed in this paper. For example, a keratocoric cornea would require a carefully chosen back surface for a comfortable fit and a precisely computed front surface for decent optical quality.

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