Uniqueness of Corneal Shape From Placido Ring Images

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Introduction.

The measurement of corneal shape has become a rapidly growing enterprise in the past few years. Corneal shape is typically measured using the Placido ring technology. A series of concentric rings (the Placido rings) are flashed, the rays reflected off the cornea are captured by a CCD camera, and the data are analyzed to reconstruct the corneal shape. All present commercially available algorithms do the reconstruction one meridian at a time and make the assumption that rays lie in the meridional plane (van Saarloos & Constable, 1991). This assumption is equivalent to assuming the corneal normal lies in the meridional plane. This assumption is not correct for real corneas since astigmatism or other deviations from axial symmetry are typically present. When the normal is allowed to deviate from the meridional plane an ambiguity results since each CCD point could correspond to an infinity of points on the continuous ring. Therefore multiple corneal shapes seem possible (Applegate, 1992). This ambiguity has prompted the suggestion that instead of using rings as the target, checkerboard patterns or rings with gaps should be used to guarantee the uniqueness of the solution (Klein, 1993).

We have developed an algorithm (Klein, et al. 1994; Halstead, et al., 1995) that uses a two-dimensional polynomial spline to represent the corneal surface. A least squares fit is used to find the corneal shape that produces the best agreement with the Placido ring data (including proper treatment of skew rays). The question to be considered in the present paper is whether our method that assumes smoothness in two dimensions is able to eliminate the ambiguity associated with previous methods. A negative answer to this question would imply that future corneal topography machines should not use discontinuous rings as targets.

Methods.

In order to make our analysis as simple as possible, a simple geometry will be assumed:

1) The camera is assumed to be telecentric such that the chief ray in object space is parallel to the optic axis. A telecentric camera can be produced by placing an aperture stop at the focal plane of the camera lens (the image plane is behind the focal plane for nearby objects). Alternatively, if the distance from the cornea to the camera is large compared to the corneal size then the system is effectively telecentric for our purposes without the extra aperture step. The advantage of a telecentric camera is that each point on the image corresponds directly with an (x, y) point on the cornea since the light leaving the cornea is parallel.

2) The light from the target rings is "collimated". This is most easily achieved by placing the rings far away relative to the corneal size.

These two assumptions greatly simplify the calculation of corneal slant (the angle between the corneal normal and the axis). Consider, for example, a target ring that is at an angle θ relative to the optic axis as seen from the corneal center. Light from that ring will be reflected from the cornea. For all the points of the cornea where the corneal slant is θ = φ/2 the rays will leave the cornea parallel to the axis. These are the chief rays for our telecentric camera, so they are the rays defining the image. Thus for our optical system one directly knows the slant, σ, of each corneal point that corresponds to an image of one of the rings. The corneal normal can be written as:

\[ n = (\sin(\alpha) \cos(\tau), \sin(\alpha) \sin(\tau), \cos(\alpha)) \]  

where σ is the slant angle and τ is the unknown tilt angle. The challenge facing us is whether we can reconstruct the surface \( n(x, y) \) when only given σ and no information about τ since the targets are circular rings that eliminate all cues to τ.
Eq. 2 can be written in terms of derivatives of corneal position as follows:

\[ n = \frac{h_x, h_y, 1}{(1 + h_x^2 + h_y^2)^{\frac{3}{2}}} \]  

(3)

where \( h_x = \frac{\partial h}{\partial x} \), \( h_y = \frac{\partial h}{\partial y} \) and the denominator is needed to ensure the normal vector has unit magnitude. Simple trigonometry reveals that the connection between the derivatives and the slant and tilt angles are:

\[ \tan^2(\sigma) = h_x^2 + h_y^2 \]

and \( \tan(\tau) = \frac{h_y}{h_x} \)

(4)

The value of \( \sigma \) in Eq. 4 is known at the image points of the CCD array. Thus the quantity, \( h_x^2 + h_y^2 \), is also known. We will call this derivative quantity \( D(x, y) \):

\[ D(x, y) = h_x^2 + h_y^2 \]

(6)

for all the points on the cornea corresponding to a given Placido ring image.

The issue of corneal shape uniqueness addressed by this paper can be stated as: given \( D(x, y) \) at a finite number of points can we calculate \( h(x, y) \) at those points. In general, the answer is no. Between the rings there is no information regarding slope so arbitrary positional discontinuities can be present. Furthermore the tilt angle \( \tau \) can vary arbitrarily around the ring and the constraint of Eq. 6 can be maintained, thereby allowing an infinity of corneal shapes to be compatible with the CCD image. If, however, we make a reasonable assumption about the two-dimensional continuity around the ring then Eq. 6 becomes a powerful constraint and one can recover \( h(x, y) \) from information about \( D(x, y) \).

Results using a finite Taylor's series (Zernike) expansion of \( h(x, y) \)

In order to recover \( h(x, y) \) from knowledge of corneal slant, \( D(x, y) \), we need to assume the surface \( h(x, y) \) has sufficient continuity. In this paper we will impose continuity by assuming the corneal surface can be represented by a finite Taylor's expansion in Cartesian coordinates \( x \) and \( y \):

\[ h(x, y) = \sum a_{mn} x^m y^n \]

(7)

with \( 0 \leq m+n \leq k \).

The finiteness constraint on \( m+n \) forces the cornea to have high order continuity. In terms of polar coordinates this assumption is identical to the statement that there are a finite number of terms in the Zernike polynomial expansion (Webb, 1992) of corneal height, \( h \). In this paper we will stick with Cartesian coordinates because of their familiarity, but we will also refer to the Zernike expansion.

Our argument that \( h(x, y) \) can be derived from \( D(x, y) \) will be made by counting parameters. In addition to \( D(x, y) \) we will also need to know \( h(0, 0) \), the location of the axial point since the slant information, \( D \), has lost all information about the absolute position of the cornea.

The first few terms of Eq. 7 are:

\[ h(x, y) = a_{00} + a_{10} x + a_{01} y + a_{20} x^2 + a_{11} xy + a_{02} y^2 + a_{30} x^3 + \cdots \]  

(8)

We have the freedom to choose the location of the origin of the coordinate system. We will choose the origin to be the corneal apex so that \( a_{00} \) and \( a_{01} \) vanish. The vanishing of \( a_{10} \) and \( a_{11} \) is achieved by choosing the \( z \) axis to intersect the cornea at the point at which the slant \( D \) vanishes. The coefficient \( a_{00} \) can be set to zero since we are free to choose \( h(0,0) \). With this choice of the coordinate origin, Eq. 8 becomes:

\[ h(x, y) = a_{20} x^2 + a_{11} xy + a_{02} y^2 + a_{30} x^3 + \cdots \]

(9)

If the upper limit to the expansion has \( k=3 \) then as seen in Eq. 9 there are \( 7 \) terms in the expansion. In general, the number of terms in the expansion of \( h \) is:

\[ N_h(k) = (k + 1)(k + 2)/2 - 3 \]

(10)

We can do a similar counting of the number of parameters of \( D \):

\[ N_D(k) = (2k)(2k + 1)/2 - 3 \]

(11)

The values of \( N_h \) and \( N_D \) for several values of \( k \) are:

<table>
<thead>
<tr>
<th>( k )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>10</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_h )</td>
<td>3</td>
<td>7</td>
<td>12</td>
<td>18</td>
<td>25</td>
<td>63</td>
<td>150</td>
</tr>
<tr>
<td>( N_D )</td>
<td>3</td>
<td>12</td>
<td>25</td>
<td>42</td>
<td>63</td>
<td>187</td>
<td>493</td>
</tr>
</tbody>
</table>

Table 1

The values in Table 1 show that the function \( D(x, y) \) has more than enough information to recover all the coefficients of \( h(x, y) \). For a quadratic surface \( (k=2) \) there are three coefficients for both \( h \) and \( D \) so there is a
surface h for every quadratic surface D. For higher order surfaces (k>2) there is a limited
class of derivatives, D(x, y) that correspond to
terms. The constraints on D are
general terms, and the constraints on D are
useful when the data is noisy since a least
squares fit could be used to extract the surface
parameters. Consider the case of k=7. The raw
test data given by D(x, y) would be
characterized by 63 parameters (63 Zernike
coefficients). These 63 data points would be fit
using 25 underlying parameters for specifying
h(x, y). Since there are substantially fewer
parameters than data, there is substantial robustness to the presence of noise.

In actuality, many more data points
specifying D(x, y) are measured than are
needed. Videokeratographs measure a
maximum of 180 meridia and 16 rings. Thus
there are more than 2880 data points that are
available for determining the 150 Zernike
coefficients for k = 16.

Eq. 6 shows that there is a sign ambiguity
in the connection between D and h. One can
reverse the signs of all the coefficients a_{mn}
without changing D. For the case of the
cornea, this ambiguity is resolved by having
the coefficients a_{00} and a_{02} be positive, since
the cornea is known to have positive curvature.

Discussion.

After making a number of assumptions we
have shown that the corneal shape is uniquely
determined from the slant information that is
available in the Placido ring image. This result
is somewhat surprising because with ring
targets it is never clear which point on the
target corresponds to which point on the image
(the raw data does not specify corneal tilt). One
might think that small shifts in the map that
connects a target point to an image point would
plausibly lead to a different corneal shape that
is fully compatible with the original reflection
data. What we have shown is that for a cornea
that is a finite power series expansion in x and
y (or equivalently a finite Zernike expansion)
there can only be a single corneal shape leading
to a given image.

This result has important implications for
the design of videokeratoscopes. A number of
researchers (Klein, 1993) have advocated VK
targets that are checkerboards rather than rings.

Our results indicate that circular rings are quite
sufficient for obtaining an unambiguous shape.

Before accepting our finding too quickly
we must be reminded that the finding of
uniqueness was based on two assumptions.
Here we examine these assumptions:

1) Telecentric camera and distant rings. The
camera and the rings were assumed to be far
from the cornea as compared to the size of the
cornea. This assumption was offered merely as
a convenience to simplify the calculation of
corneal slant given the reflected image of the
Placido rings. There is a straightforward
method to deal with the case in which the
corneal size can not be neglected (Klein, 1992;
clear that our uniqueness argument should not
change in this more realistic case, but it would
be nice to see a mathematical proof.

2) Finite Taylor's (Zernike) series. The critical
assumption of our derivation is that there is a
patch of cornea that included the origin for
which D(x, y) has a finite Zernike expansion.
This is our main assumption. If it were
removed then there would no longer be
uniqueness. Corneal shape would be
ambiguous since multiple corneal shapes could
produce identical image rings. The assumption
of a finite Zernike series is however, not
unreasonable since corneal surfaces are
relatively smooth. It would be nice to see in
what ways this assumption could be relaxed
and still maintain uniqueness.

We have carried out a number of computer
simulations which provide further evidence
that our assumptions are reasonable. We used
ellipsoidal corneal shapes that included torticity
(astigmatism). The ring and camera locations
were close to the cornea in order to simulate a
real videokeratoscope. A biiquintic spline was
used to represent corneal shape. A least
squares fit was done in order to find the shape
that produced the best match to the image rings
(Halstead, et al. 1994). In all our simulations
we always found a unique solution. The
solution agreed with the original ellipsoidal
shape to within 10^-2 microns. The derivation
provided by the present paper explains why a
unique surface was found.

Dealing with radial keratotomy (RK). Ray
Applegate (personal communication) has
pointed out that patients with RK present a
special problem for Placido ring algorithms because there can be discontinuities across the RK cuts. Each of the plates between the cuts can have an arbitrary tilt that could be difficult to recover, since as we have discussed, the use of Placido ring targets eliminates tilt information from the raw data. However, we believe that even in the extreme case of RK a robust recovery of \( h(x, y) \) is possible based on Placido ring targets. There are two approaches for recovering \( h \). The simple approach would be proceed as if there were no discontinuities across the RK cuts. With \( k=16 \) (150 Zernike coefficients) there is enough freedom to approach the corneal shape fairly well. The value of \( k=16 \) was chosen so that the angular Zernike polynomials would include terms like \( \cos(16\theta) \) which is the second harmonic of \( \cos(8\theta) \) that represents the repetition corresponding to the 8 cuts of some RK procedures.

An approach that would produce more accurate fits to the RK corneas would be to use a Zernike fit for the central region of the cornea and 8 splined regions for the outer plates (for the case of 8 cuts). In this situation there would be more parameters to fit as compared to the non-RK case, but the number of data is still large enough that there should not be a problem in demonstrating uniqueness and in estimating the parameters. The eight splines would allow discontinuous height across the plates but continuity would be enforced everywhere else.

Implications. A proof of uniqueness has important implications for the design of future videokeratographs. Based on the ambiguity of which point on the target ring connects to which point on the image ring, several researchers have suggested that two-dimensional targets such as radial checkerboards should replace the rings. Our finding is that two-dimensional targets may not be needed as long as there is sufficient continuity of the corneal shape. Two-dimensional targets would, however, be useful for speeding up our spline-based least squares algorithm since it would enable us to make improved estimates of our first guess for starting the least squares search. An argument against a two-dimensional target is that it could interfere with the edge finding algorithm that is used to accurately locate the image ring positions. It is crucial to estimate the rings very accurately and any noise is unwanted.

The algorithm discussed in this paper for recovering corneal shape has the important feature that it does not make the common, faulty assumption that the corneal normal lies in the meridional plane. Skew rays are treated correctly.

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References.


